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# ANALYSIS

# ÆQUATIONUM.

AUCTORE William (GUIL.) HALES, D.D.

COLL. S. TRIN. DUBLIN. SOCIO.

DUBLINII:

EXCUDEBAT JOSEPHUS HILL.

MDCCLXXXIV.

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P R Æ P O S I T O

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# P R Æ F A T I O.

*ÆQUATIONUM Theoria, utpote permagni in matheſi momenti, trecentos fermè per annos, a tempore* Lucæ de Burgo *ad uſque præſens, Analyſtarum ingenia exercuit. Veruntamen fatendum eſt, quod Analyſtæ maximi, inventis ſuis algebræ fines extendere utplurimùm ſatagunt, de inventorum evidentiá demonſtrationibus validis atque legitimis ſtabiliendá parùm ſolliciti; "quoniam hæ ſatis faciles ſibi viſæ ſunt, & nonnunquam abſque nimiis ambagibus tradi non poſſent"—quales memorat cauſas* Newtonus, *in* Arithmetica Univerſali, *cur, æquationum proprietates docens, demonſtrationes non ſemper adjunxit. Hanc itaque curam poſteris legant. Inde vero fit, quod regulas æquationum præſtantiſſimas, vel imperfecte tradunt, vel quaſi demonſtrationibus idoneis deſtitutas omittunt omnes fere algebræ inſtitutiones. Sic, ut pauca exempla e multis proferam,* Newtoni *celeberrimum theorema binomium, necnon germanum illud de inveniendis ſummis poteſtatum radicum æquationis; vel nullis, vel minus accuratis, vel denique alieno ex fonte petitis demonſtrationibus, omnibus fere in libris elementariis extant:* Carteſii *theorema egregium de numero radicum affirmativarum & negativarum æquationis dignoſcendo, plerique abſque demonſtratione tradunt, nonnulli negligunt: Methodus vero detegendi radices æquationum impoſſibiles, ſi quæ ſint, earumque numerum, exceptá regulá quadam imperfectá a*

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Newtono *traditâ, ex omnibus exulat. Hunc igitur defectum, aliquatenus ſaltem, ſupplere, noſtri inſtituti eſt ratio; ne ſcientia præſtantiſſima quidem, & quæ ipſi geometriæ vix forſan evidentiâ cederet, ſi modo lucido ordine & methodo veterum Geometrarum inſiſterent Analyſtæ, falſo incertitudinis crimine premi videatur.* Newtoni Arithmeticam Univerſalem *potiſſimum, ut interpres ſecutus ſum, in conſilium ſubinde adhibitis, peritiſſimis in re mathematicâ ingeniis,* Wallis, McLaurin, Saunderſon, De Moivre, Simpſon, Clairaut, D'Alembert, Euler, La Grange, Waring, Bertrand, Landen, Hutton, *&c. &c. qui aut ſcriptis* Newtoni *illuſtrandis, aut algebræ limitibus latius proferendis feliciſſime operam dederunt. Curſim quoque adſtruitur brevis hiſtoria præcipuarum inventionum analyticarum, ad levandum diſquiſitionis abſtruſioris tædium; & notantur auctores quorum ex ſcriptis pleniorem harum rerum cognitionem haurire licet. Quid ipſe præſtiterim aliorum eſto judicium. Hoc ſaltem præmonitos velim lectores, quod opus elementarium methodo elegantiſſimâ abſolvere, analyſi breviſſimâ & ſimul perſpicuâ, ſyntheſi concinnâ & minime operoſâ, longe difficillimum eſt.* Un petit génie pourra s'imaginer qu'il eſt facile de faire un livre élémentaire clair & précis—les vrais Savans en connoiſſent la difficulté.

Sauri, Inſtitutions Mathematiques.

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# De NATURA ÆQUATIONUM.

§. 1. ÆQUATIONES ſunt quantitatum aut ſibi mutuo æqualium, aut ſimul nihilo æquipollentium congeries. Priores appellantur *canonicæ*, poſteriores *originales*.

2. Duobus modis conſiderandæ veniunt æquationes, vel ut ultimæ concluſiones ad quas in Problematis ſolvendis deventum eſt, vel ut media quorum ope finales æquationes acquirendæ ſunt. Æquatio finalis ex unica tantum quantitate incognita cognitis involuta conflatur, modò Problema definitum ſit et aliquid certi quærendum innuat. Mediæ autem, quæ problematum conditiones deſignant, plures involvunt quantitates incognitas, atque ideo debent inter ſe comparari & ita connecti ut ex omnibus una tandem emergat æquatio nova cui ineſt unica quam quærimus incognita quantitas admiſta cognitis: cujus reductione quantitatis incognitæ valor comparandus eſt.

3. Propoſita igitur quæſtione aliquâ, artificis ingenium in eo præſertim requiritur ut omnes ejus conditiones totidem æquationibus deſignet. Ad quod faciendum, perpendat imprimis an propoſitiones ſive ſententiæ quibus enunciatur ſint omnes aptæ quæ terminis algebraicis deſignari poſſint, haud ſecus quam conceptus noſtri characteribus græcis vel latinis. Et ſi ita, (ut ſolet in quæſtionibus quæ circa numeros vel abſtractas quantitates verſantur) deinde nomina quantitatibus ignotis atque etiam notis, ſi opus fuerit, imponat[a]; & ſenſum quæſtionis, ſermone, ut ita loquar,

[a] Algebriſtæ antiquiores ſolas quantitates incognitas per literas deſignarunt, notas autem per ipſos numeros, puta, A — 10 = 24. Uſum literarum alphabeti vice ſignorum antea adhibitorum primus introduxiſſe fertur *Buteo*, qui Tractatum Algebraicum edidit anno 1559. Recentiores vero poſt *Vietam* fere omnes deſignant quantitates tam datas quam quæſitas per literas aut ſpecies, cum magno artis

analytico deſignet. Et conditiones ejus ad algebraicos terminos ſic tranſlatæ, tot dabunt æquationes quot ei ſolvendæ ſufficiunt.

4. Quemadmodum *ſi quærantur duo numeri quorum ſumma ſit* 140, *et productum* 3136. Deſignantibus $x$ et $y$, numeros quæſitos, quæſtio deducitur ad æquationes $x + y = 140$, et $xy = 3136$. Quarum ope per regulas mox tradendas inveſtigandi ſunt $x$ et $y$.

5. Aut, *ſi quærantur tres numeri continue proportionales, quorum ſumma ſit* $20 = a$, *et quadratorum ſumma* $140 = b$. Sint numeri quæſiti $x$, $y$ et $z$, et ex quæſtione prodeunt æquationes $xz = y^2$, $x + y + z = a$, et $x^2 + y^2 + z^2 = b$. Quarum reductione eruendi ſunt $x$, $y$ et $z$.

6. Cæterum notandum eſt, ſolutiones quæſtionum eo magis expeditas et artificioſas utplurimum evadere quo pauciores incognitæ quantitates ſub initio ponuntur. Aliquando autem harum aliquam ex quæſtionum conditione aliqua deſignare licebit, dum ex conditionibus reliquis elicientur æquationes quæ in ſolutionem poſtulantur.

analyticæ emolumento, ſive laboris inter operandum minuendi, ſive theorematum generalium eliciendorum gratiâ. Hinc diſtinctionis ergo, forma vetuſtior appellatur Algebra *numeroſa*, recentior *ſpecioſa*. At vero *Vieta*, eumque ſecuti *Oughtredus* aliique, uſi ſunt literis majuſculis; hodierni fere omnes partim brevitatis et compendii causâ, partim ad evitandam confuſionem quæ aliter in praxi non raro extaret, ex eo quod literæ majusculæ deſignandis figurarum geometricarum lineis tribuuntur, adhibent minuſculas. Quantitates datas diſtinxit *Carteſius* per literas alphabeti primas, $a$, $b$, $c$, &c. quæſitas autem per ultimas $z$, $y$, $x$, &c. Quantitatum poteſtates primus quoque deſignavit indicibus integris $a^2$, $a^3$, $a^4$, &c. Quantitatum radices deſignaſſe fertur *Albertus Girardus* indicibus fractis, $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, &c. pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c:a^5}$, &c. Quantitates reciprocas indicibus negativis notavit *Walliſius*, pro $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{a^3}$, &c. ſcribendo $a^{-1}$, $a^{-2}$, $a^{-3}$, &c. qui et uſum indicum indefinitorum et ſurdorum introduxit. Cæterum ex quo invaluit indicum numeralium uſus, ut evitetur confuſio, coefficientes numerales literis præfixæ ſunt, quas affigebant vetuſtiores ſcribendo nimirum $x3$ pro $3x$.

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Sic in quæſtione priore, ex una conditione quod ſumma quantitatum incognitarum ſit 140, unam deſignante $x$, alteram deſignabit $140 - x$; ex altera conditione autem elicietur æquatio $x \times \overline{140 - x} = 3136$, cujus reductio dabit $x$. Atque in quæſtione altera, ex prima conditione quod numeri quæſiti ſint continue proportionales, exponentibus $x$ et $y$, primum et ſecundum, exponet $\frac{y^2}{x}$, tertium; ex reliquis autem conditionibus prodeunt duæ æquationes, $x + y + \frac{y^2}{x} = a$ et, $x^2 + y^2 + \frac{y^4}{x^2} = b$. Quarum reductione $x$, et $y$ determinandi ſunt. Interdum etiam plures vel omnes quantitates incognitas ex conditionibus quæſtionum deſignare fas erit. Ut ex ſequentibus patebit.

7. Etſi deſideratas quantitates utplurimum immediate quærere ſolemus, ſi quando tamen duæ obvenerint ambiguæ, hoc eſt, quæ conditionibus omnino ſimilibus præditæ ſunt (ut quatuor numerorum continue proportionalium duo medii et duo extremi) præſtat alias quantitates non ambiguas quærere, quas inter et haſce intercedit nota quædam relatio, quemadmodum harum ſummam vel differentiam, vel rectangulum.

8. Nonnumquam quæſtiones geometricæ eadem facilitate iiſdemque legibus ad æquationes redigi poſſunt ac quæ de abſtractis quantitatibus proponuntur, ubi ſcilicet ſolæ quantitatum proportiones abſque poſitionibus linearum conſiderandæ veniunt. Ut *ſi data quædam recta in media et extrema proportione ſecanda ſit*; deſignante $a$ rectam datam, et $x$ ſegmentum majus, erit $a - x$ ſegmentum minus, et $x^2 = a \times \overline{a - x}$; æquatio quæ per reductionem dabit $x$.

9. Quod ſi ſermo quocum ſtatus quæſtionis exprimitur ineptus videatur qui in algebraicum poſſit verti, paucis mutationibus adhibitis, et ad ſenſum potius quam verborum

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ſonos attendendo expedietur verſio. Cæterum methodum redigendi hujuſmodi problemata ad æquationes certis legibus vix inſtituere fas eſt, cum ex proprio cujuſque analyſtæ ingenio plurimum pendeat; exemplis ſane et exercitatione facilius quam præceptis illuſtranda et addiſcenda eſt. Placet igitur jam paucas inſtantias adjungere.

10. *Mercator quidam nummos ejus triente quotannis adauget, demptis £100 quas annuatim impendit in familiam, et poſt tres annos fit duplo ditior; quæruntur nummi?*

Plures latent in hac quæſtione propoſitiones, quæ omnes ſic eruuntur et enunciantur.

| | |
|---|---|
| Mercator habet nummos quoſdam - - - - - - - | $x$ |
| Ex his anno primo expendit £100 - - - - | $x - 100$ |
| Et reliquum adauget triente - - - - - - - | $x - 100 + \frac{x - 100}{3}$ ſive $\frac{4x - 400}{3}$ |
| Annoque ſecundo expendit £100 - - - | $\frac{4x - 400}{3} - 100$ ſive $\frac{4x - 700}{3}$ |
| Et reliquum adauget triente - - - | $\frac{4x - 700}{3} + \frac{4x - 700}{9}$ ſive $\frac{16x - 2800}{9}$ |
| Et ſic anno tertio expendit £100 - | $\frac{16x - 2800}{9} - 100$ ſive $\frac{16x - 3700}{9}$ |
| Et reliquum adauget triente - | $\frac{16x - 3700}{9} + \frac{16x - 3700}{27}$ ſive $\frac{64x - 14800}{27}$ |
| Fitque duplo ditior quam ſub initio - | $\frac{64x - 14800}{27} = 2x$. |

Quæſtio itaque ad æquationem $\frac{64x - 14800}{27} = 2x$ perducitur, cujus reductione eruendus eſt $x$.

11. Ex epitaphio quodam Arithmetico quæritur ætas *Diophanti* celeberrimi analyſtæ *Alexandrini* qui Anno 365 circiter floruit, et primus extat Algebræ ſcriptor apud Græcos.

Οὗτός τοι Διόφαντον ἔχει τάφος, ἆ μέγα θαῦμα·
Καὶ τάφος ἐκ τέχνης μέτρα βίοιο λέγει.
Ἕκτην κουρίζειν βιότου Θεὸς ὤπασε μοίρην,
Δωδεκάτῃ δ' ἐπιθεὶς μῆλα πόρεν χλοάειν.

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Τῇ δ' ἄρ ἐπ' ἑβδομάτῃ τὸ γαμήλιον ἥψατο φέγγος,
Ἐκ δὲ γάμων πέμπτῳ παῖδ' ἐπένευσεν ἔτει.
Αἴ αἴ τηλύγετον δειλὸν τέκος, ἥμισυ πατρὸς,
Τοῦ δὲ καὶ ἡ κρυερὸς μέτρον ἑλὼν βιότου.
Πένθος δ' αὖ πισύρεσσι παρηγορέων ἐνιαυτοῖς
Τῇ δὲ πόσου σοφίῃ τέρμ' ἐπέρησε βίου;

Indicat utique epitaphium, *Diophantum* ſextam ætatis partem egiſſe puerum; partem inde duodecimam, puberem; completa ætatis parte ſeptima poſt, uxorem duxiſſe; interjecto autem quinquennio, filium genuiſſe; qui ſemiſſem paternæ ætatis quum attigiſſet, diem obiit ſupremum; patrem vero quadriennio poſt, fato quoque ceſſiſſe. Quæritur igitur *Diophanti* ætas: ſive in genere, quæritur numerus cujus partes ſexta, duodecima, ſeptima adjuncto quinario, et ſemiſſis adjuncto quaternario, ſimul conficiant totum. Sit numerus quæſitus $x$, et habebitur æquatio $\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4 = x$. Cujus reductione eruendus eſt $x$.

12. Quod ſi quæſtio ipſa omnes conditiones quæ in ſolutionem poſtulantur non ſuppeditet, evolvendæ ſunt quantitatum tum datarum tum quæſitarum relationes, quoquo modo compertæ, ſi quæ aptæ ſint equationibus rite pariendis. Hoc pacto etenim problemata, quæ primo conſpectu inſolubilia videntur, præſertim in *phyſicis* vel *mixtâ matheſi*, haud raro ſolutionem ſubire coguntur. Sic opificis fraudem, qui furatus eſt partem aliquam auri quod coronæ votivæ conflandæ gratiâ ei commiſerat *Hiero*, et argento ſubſtituto coronam juſti ponderis fabricavit, neutiquam detexiſſet *Archimedes*, rege jubente, ex datis tantum coronæ ipſius pondere et mole, ni pondera ſpecifica auri et argenti inſuper in computum adſciverat[b]. Uti poſthac oſtendetur in reductione problematum indeterminatorum.

[b] Vide *Vitruvium*, lib. 9, c. 3.



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13. Si ex quæſtione utcunque eliciendæ ſunt tot æquationes quot habet incognitas quantitates, determinata erit, et ſolutionem præbebit. Quod ſi conditionibus deficientibus, pauciores dentur æquationes quam incognitæ quantitates, quæſtio indeterminata erit, vel quodam reſpectu vel abſolute; et modò complures, modò innumeras ſortietur ſolutiones. Ut *ſi quærantur duo numeri* x *et* y *quorum ſumma ſit* 10. Deſignante $x$, 9; erit $y = 1$, deſignante $x$, 8; erit $y = 2$, &c. et inde novem prodeunt ſolutiones, ſi de numeris integris et poſitivis agatur; innumeræ autem, ſi de fractis vel negativis. Contra, ſi redundantibus conditionibus, plures habentur æquationes quam incognitæ quantitates, quæſtio quaſi pluſquam determinata evadet, et concluſio erit abſurda ſi inter ſe diſcordent conditiones, aut inepta ſi non. Sic *quæſitis duobus numeris quorum ſumma ſit* 8, *differentia* 2, *et productum* 12. Sint numeri $x$, et $8 - x$; eritque $x - 2 = 8 - x$. Adeoque $x = 5$, et alter $8 - x = 3$. Sed per tertiam conditionem productum $5 \times 3$ ſeu $15 = 12$ quod abſurdum. Et ſi hæc fuiſſet conditio ut *productum ſit* 15; inutilis evaderet, quippe quæ ex cæteris pendet, et æquationem identicam ſeu ineptam $15 = 15$ parit. Atque hoc pacto ſane detegentur quæſtionum conditiones aut ab invicem pendentes, aut inter ſe diſcordantes; ubi ſcilicet, opere rite inſtituto, ad æquationem identicam vel abſurdam devenitur.

## De Dimensionibus Æquationum.

14. Dimenſiones æquationum ad quas in problematis ſolvendis deventum eſt, ex maxima dimenſione quantitatis incognitæ æſtimantur, nullo reſpectu ad quantitates cognitas habito, nec ad terminos intermedios. Sic æquatio $\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4 = x$ eſt ſimplex ſeu primi gradus, $140x - x^2 = 3136$ quadratica ſeu ſecundi gradus, & ſic deinceps. Et ſi æquatio plures quantitates incognitas ſortiatur, gradus hujus perinde quoque æſtimantur ex

maxima dimenſione ejuſdem aut plurium quantitatum incognitarum. Sic $xy = 3136$, et $x^2 + y^2 + \frac{y^4}{x} = b$ ſunt quadraticæ, $xyz + x^2 = a$ cubica, &c. Problema inſequens ad æquationes quotcunque dimenſionum deducet.

15. *Vir quidam nummos* a *in mercaturam impendit, quam vero hâc non ſuccederet, ſe expedire cupit ſub fine anni primi; at tali occaſione non oblatâ, ſolvi nequit niſi ſub fine anni ſecundi, aut tertii, aut denique anni* n*; quo tempore, ex nummis poſt primi anni jacturam redeuntibus deperit* b. *Quæritur damnum ejus pro ſingulis* £100 *quotannis?*

Sit $x$ damnum quæſitum, et inſtituendo analogiam, $(100 : 100 - x :: a :)$ $a \times \frac{100 - x}{100}$ ſive $a \times \overline{1 - \frac{x}{100}}$ exponet mercatoris nummos poſt primi anni jacturam; et continuando analogiam, $(100 : 100 - x :: a \times \frac{100 - x}{100} :)$ $a \times \overline{\frac{100 - x}{100}}\Big|^2$ ſive $a \times \overline{1 - \frac{x}{100}}\Big|^2$ nummos poſt ſecundi anni jacturam; ſimiliter, $a \times \overline{1 - \frac{x}{100}}\Big|^3$ nummos poſt anni tertii jacturam; et in genere, $a \times \overline{1 - \frac{x}{100}}\Big|^n$ nummos poſt anni $n$ jacturam. Exponet item $a \times \overline{1 - \frac{x}{100}} - b$ mercatoris nummos ſub fine anni cujuſque poſt primum. Poſito igitur illum a mercatura ſolvi ſub fine anni ſecundi, æquatio quadratica $a \times \overline{1 - \frac{x}{100}}\Big|^2 = a \times \overline{1 - \frac{x}{100}} - b$, per reductionem dabit $x$; ſub fine anni tertii, æquatio cubica $a \times \overline{1 - \frac{x}{100}}\Big|^3 = a \times \overline{1 - \frac{x}{100}} - b$; et ſub fine anni $n$, æquatio $n^{ti}$ gradus $a \times \overline{1 - \frac{x}{100}}\Big|^n = a \times \overline{1 - \frac{x}{100}} - b$, per reſolutionem dabit $x$.

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16. His præfatis de natura et dimenſionibus æquationum ad quas problemata deduci debent, inſtituenda eſt earum analyſis. Trademus itaque modos,

I. Concinnandi æquationes quotcunque dimenſionum; et reducendi quadraticas, aliaſque ordini quadraticarum adſcribendas.

II. Ex mediis pluribus in ſe collatis eliciendi finales æquationes, et reducendi indeterminatas primi et ſecundi gradus.

III. Reſolvendi finales æquationes quotcunque dimenſionum.

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# ANALYSIS ÆQUATIONUM.

## PARS PRIMA.

### De Concinnanda Æquatione Solitaria.

17. Quoniam quantitates incognitæ quas referunt æquationes cognitis involvuntur per varios modos additionis vel ſubductionis, multiplicationis vel diviſionis, involutionis vel evolutionis; neceſſe eſt, ut per operationes iis contrarias quibus cum cognitis implicitæ fuerunt, extricentur et expediantur. De ordinatione & reductione igitur æquationis ſolitariæ in ſequentibus regulis agetur; quæ hoc pendunt principio, quod ſi quantitatibus quibuſlibet ſibi mutuo æqualibus addantur vel auferantur æquales, multiplicentur vel dividantur per æquales, involvantur vel evolvantur æqualiter, manebit adhuc quantitatum per has operationes reſultantium æqualitas.

18. Reg. I. *Termini æquationum ſecundum dimenſiones incognitæ quantitatis in ordinem ſemper redigendi ſunt.* Ita ſcilicet ut primum occupent locum in quibus incognita quantitas eſt plurimarum dimenſionum, et ſecundum locum in quibus ea eſt una dimenſione minor, et ſic deinceps; ad normam ſequentium: in quibus $x$ deſignat quantitatem quæſitam; $p$, $q$, $r$, $s$, alias quaſcunque quantitates ex quibus determinatis et cognitis etiam $x$ determinatur, et per regulas inſequentes erui poteſt valor ejus ſi ſit unius aut duarum dimenſionum, ſive maximæ poteſtatis ejus ſi plurium.



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| | |
|---|---|
| $x = p;$ | vel, $x - p = 0,$ |
| $x^2 = px + q,$ | $x^2 - px - q = 0;$ |
| $x^3 = px^2 + qx + r;$ | $x^3 - px^2 - qx - r = 0;$ |
| $x^4 = px^3 + qx^2 + rx + s,$ &c. | $x^4 - px^3 - qx^2 - rx - s = 0,$ &c. |

19. Reg. II. *Siquæ ſunt quantitates quæ ſe mutuo deſtruere, vel per additionem aut ſubductionem coaleſcere poſſunt, termini perinde minuendi ſunt.*

Sic § 11, æquatio $x = \frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4$, addendo partes $x$ et numeros integros reſpective, prodit $x = \frac{75x}{84} + 9$, et ſubducendo $\frac{75x}{84}$ utrobique, $\frac{9x}{84} = 9$. Sic $a^2 + x = a^2 + b$, deletâ $a^2$ per ſubductionem, dat $x = b$. Sic quoque $\sqrt{48} = x - \sqrt{75}$, dat $x = \sqrt{48} + \sqrt{75} = 4\sqrt{3} + 5\sqrt{3} = 9\sqrt{3}$. Itemque $\sqrt{48a^2bc} - x = \sqrt{\frac{16}{27}a^2bc}$ dat $x = \sqrt{48a^2bc} - \sqrt{\frac{48}{81}a^2bc} = 4a\sqrt{3bc} - \frac{4}{9}a\sqrt{3bc} = \frac{32}{9}a\sqrt{3bc}$. Ad hanc igitur regulám referri debet ordinatio terminorum æquationis quæ fieri ſolet per tranſlationem ad partes contrarias cum ſigno mutato.

20. Reg. III. *Siqua compareat quantitas per quam omnes æquationis termini multiplicantur, debent omnes per illam dividi; vel ſi per eandem quantitatem omnes dividantur, debent omnes per illam multiplicari.*

Sic $15b^2 = 24ab + 3bx$, dividendo per $3b$ et tranſponendo, prodit $x = 5b - 8a$. Itemque, $\frac{x}{c} - \sqrt{\frac{ab^3}{c^2}} = \frac{1}{2c}\sqrt{a^3b - 4a^2b^2 + 4ab^3}$, multiplicando per $c$, et tranſponendo, dat $x = \sqrt{ab^3} + \frac{1}{2}\sqrt{a^3b - 4a^2b^2 + 4ab^3} = b\sqrt{ab} + \frac{1}{2} \times \overline{a - 2b}\sqrt{ab} = \frac{1}{2}a\sqrt{ab}$.

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21. *Aliquando reductio inſtitui poteſt dividendo æquationem per compoſitam aliquam quantitatem.* Sic $y^3 = \overline{b - 2c}\, y^2 + 3bcy - b^2c$, transferendo omnes terminos ad eandem partem, et dividendo per $y - b$, dat $yy + 2cy - bc = 0$. De inventione hujuſmodi diviſorum perutili poſthac agemus.

22. *Hoc pacto tolli poſſunt fractiones ex æquationibus.* Multiplicando ſcil; omnes terminos per uniuſcujuſque denominatorem. Sic habitâ § 10, æquatione $\frac{64x - 14800}{27} = 2x$, multiplicando per 27, et tranſponendo, prodit $10x = 14800$.

23. REG. IV. *Si maxima poteſtas quantitatis incognitæ multiplicetur vel dividatur per cognitam quamlibet quantitatem, debet tota æquatio per eandem dividi vel multiplicari.*

Sic æquatio noviſſima $10x = 14800$, dividendo per 10, dat $x = 1480$, qui utique exponet *Mercatoris* nummos § 10, ut et lucrum. Itemque æquatio § 19, $\frac{9x}{84} = 9$, dividendo per $\frac{9}{84}$ fit $x = 84$. Vixit igitur *Diophantus* 84 annos. Et ſi habeatur $\sqrt{18ab^3}\, x = 6ab^2$, concinnando prodit $x = \sqrt{\frac{36a^2b^4}{18ab^3}} = \sqrt{2ab}$. Sic § 17, æquatio quadratica $a \times \left(1 - \frac{x}{100}\right)^2 = a \times \overline{1 - \frac{x}{100}} - b$, quadrando $1 - \frac{x}{100}$, et ordinando, prodit $x^2 - 100x = -10000\,\frac{b}{a}$. Et æquatio cubica $a \times \overline{1 - \frac{x}{100}}^3 = a \times \overline{1 - \frac{x}{100}} - b$, concinnando prodit $x^3 - 300x^2 + 20000x = 1000000\,\frac{b}{a}$, et ſic deinceps.

24. Reg. V. *Siqua ſit fractio irreducibilis in cujus denominatore reperiatur incognita quantitas, omnes æquationis termini per iſtum denominatorem, aut per aliquem diviſorem ejus multiplicandi ſunt.*

Sic æquatio $\frac{ax}{a-x}+b=x$, multiplicando per $a-x$ et tranſponendo prodit $x^2=bx-ab$. Et $\frac{a^2b^2}{cx^2}=\frac{x^2}{a+b-x}$ multiplicando primo per $x^2$, deinde per $a+b-x$ evadit $\frac{a^3b^2+a^2b^3-a^2b^2x}{c}=x^4$.

25. Reg. VI. *Sicui ſurdæ quantitati irreducibili involvatur quantitas incognita, cæteri omnes termini ad partes contrarias transferendi ſunt, et utraque æquationis pars in ſe* ſemel *multiplicanda ſi radix* quadratica *ſit, vel* bis *ſi ſit* cubica, *&c.*

Sic $\sqrt{a^2-ax}+a=x$ tranſlata $a$ ad alteram partem fit $\sqrt{a^2-ax}=x-a$, et quadratis partibus, $a^2-ax=x^2-2ax+a^2$, et concinnando, $x=a$. Sic etiam $\sqrt{3:\overline{a^2x+2ax^2-x^3}}=a-x$, partibus cubice multiplicatis evadit $a^2x+2ax^2-x^3=a^3-3a^2x+3ax^2-x^3$, ſeu $x^2=4ax-a^2$.

26. Reg. VII. *Aliquando reductio per extractionem radicis ex utraque æquationis parte inſtituitur.*

Quemadmodum ſi habeatur æquatio pura $x^2=\frac{1}{4}a^2-b^2$, extracta utrobique radice prodit $x=\sqrt{\frac{1}{4}a^2-b^2}$. Quod ſi habeatur $x^2+a^2=2ax+b^2$, transfer $2ax$, et exurget quadratum ex una parte perfectum $x^2-2ax+a^2=b^2$, extractiſque partium radicibus, $x-a=b$, ſeu $x=a+b$. Sic etiam $x^3-12x^2+48x=98$, demendo utrinque 64, exhibet cubum ex altera parte perfectum, ſcil; $x^3-12x^2+48-64=34$, extractiſque radicibus cubicis partium, prodit $x-4=\sqrt[3]{34}$ ſive $x=4+\sqrt[3]{34}$.

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27. Cæterum æquationes ad quas in problematis ſolvendis deventum eſt, raro puras aut perfectas poteſtates exhibent ex iſta æquationis parte ad quam transferendi ſunt termini qui quantitatem incognitam aut ejus dimenſiones continent, quales jam poſuimus. Formam utplurimum imperfectam induunt, deficientibus nimirum uno pluribuſve terminis qui quadrato, aut cubo, &c. complendo inſerviunt, ut § 23, æquatio quadratica $x^2 - 100\,x = -10000\,\frac{a}{b}$, et cubica $x^3 - 300\,x^2 + 20000\,x = 1000000\,\frac{a}{b}$. Methodum generalem quidem reducendi æquationes hujuſmodi quadraticas, quas *adfectas* vocant, (quia præter quadratum incognitæ quantitatis, ſimplici quoque ejus poteſtate adficiuntur) jam docebimus; verùm methodus generalis reſolvendi æquationes altiorum dimenſionum per evolutionem, adhuc deſideratur.

## De Reductione Æquationum Quadraticarum.

28. Si habeatur æquatio pura $x^2 = q$, (ubi $q$ deſignat quantitatem quamcumque cognitam, ſive ſimplex fuerit ſive compoſita) extracta utrinque radice prodit $\pm x = \pm \sqrt{q}$, ſive, ſi explicetur hæc forma compendiaria, $+x = +\sqrt{q}$, $+x = -\sqrt{q}$, $-x = +\sqrt{q}$, vel $-x = -\sqrt{q}$. At ex his quatuor valoribus $x$, tertius tranſponendo revocatur ad ſecundum, et quartus ad primum; ſufficit igitur ſi affirmative tantum ponatur $x$, et bini ſtatuantur valores ejus, ſcil; $+\sqrt{q}$ vel $-\sqrt{q}$. Quod in ſequentibus quoque obſervandum. Incognitam vero quantitatem $x$, extracta radice quadraticâ duplicem ſortiri valorem, æquum eſt; quoniam enim quantitatis ſive affirmativæ ſive negativæ quadratum eſt ſemper affirmativum, e converſo, quadrati affirmativi radix poteſt eſſe



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affirmativa vel negativa. 2do. Sit $x^2 = -q$, uterque valor $x$, ſcil; $+\sqrt{-q}$, vel $-\sqrt{-q}$ prodit impoſſibilis; cum quadrati negativi geneſis ſit impoſſibilis.

Hinc in æquatione quadratica pura, binæ exiſtunt radices, ſeu bini valores quantitatis incognitæ, quæ evadent ſimul reales ſi $q$ ſit affirmativa, et ſimul impoſſibiles ſi $q$ ſit negativa.

29. Quod ſi habeatur æquatio adfecta $x^2 + px = q$, (deſignantibus $p$ et $q$ quantitates quaſcunque cognitas) conſtat, membrum $x^2 + px$, a parte ſiniſtra æquationis, imperfectum exhibere quadratum binomiæ $x + \frac{1}{2}p$; quæ utique quadrando parit $x^2 + px + \frac{1}{4}p^2$. Utrinque igitur, a parte ſiniſtra ad complendum quadratum, et a parte dextra ad ſervandam membrorum æqualitatem addatur $\frac{1}{4}p^2$, (quarta pars quadrati coefficientis ſecundi termini,) et prodibit $x^2 + px + \frac{1}{4}p^2 = \frac{1}{4}p^2 + q$, et extracta utrinque radice, $x + \frac{1}{2}p = \pm\sqrt{\frac{1}{4}p^2 + q}$, et tranſponendo, $x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + q}$. Et ſic univerſaliter: *Si ſit* $x^2 = \cdot px \cdot q$, *erit* $x = \cdot \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 \cdot q}$. *Ubi* ½ p & q *iiſdem ſignis ac* p & q *in æquatione priore afficienda ſunt, ſed* ¼ p² *ſemper affirmative ponendum.* Eſtque hoc exemplum Regula ad cujus ſimilitudinem æquationes omnes quadraticæ adfectæ ad formam ſimplicium reduci poſſunt.

30. Hinc, variantibus ſignis $p$ et $q$, quatuor prodeunt quadraticarum formulæ [b]. Scilicet,

[c] Hæc methodus complendi quadratum in ſpeciebus ad quadraticarum reductionem, extat in opere primi ſcriptoris algebraici *Luca de Burgo*, cui titulus *Summa de Arithmetica et Geometria*, impreſſo, Anno 1494, atque iterum Anno 1523. Hic tres formulas priores inſequentibus verſibus ſemibarbaris expoſuit: in quibus *res* et *radix* deſignant $px$; *cenſus*, $x^2$; *numerus* et *drachma*, $q$.

I. $x^2 + px = +q$ adeoque $x = -\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q}$ vel, $= -\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q}$

II. $x^2 - px = +q$ . . $x = +\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q}$ . $= +\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q}$

III. $x^2 - px = -q$ . . . $x = +\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$ . $= +\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$

IV. $x^2 - px = -q$ . . $x = -\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$ . $= -\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$

In formulis I et II, ubi quantitas abſoluta $q$ eſt affirmativa, erit $\sqrt{\frac{1}{4}p^2 + q}$ major quam $(\sqrt{\frac{1}{4}p^2} =) \frac{1}{2}p$, et proinde radix prior in utraque erit affirmativa, et poſterior negativa.

In formulis III et IV, ubi $q$ eſt negativa, ſi ſit *minor* quam $\frac{1}{4}p^2$, erit $\sqrt{\frac{1}{4}p^2 - q}$ affirmativa, et minor quam $\frac{1}{2}p$; et proinde in III, evadet utraque radix affirmativa, in IV, utraque negativa. Si ſit *æqualis* $\frac{1}{4}p^2$, evaneſcet $\sqrt{\frac{1}{4}p^2 - q}$, et binæ radices in utraque evadent æquales, ſeu coaleſcent in unam. Et ſi $q$ ſit *major* quam $\frac{1}{4}p^2$, erit $\frac{1}{4}p^2 - q$ negativa, et $\sqrt{\frac{1}{4}p^2 - q}$ impoſſibilis; et proinde binæ radices in utraque formula evadent impoſſibiles, quoniam hæc quantitas impoſſibilis earum compoſitionem ingreditur.

1. *Si res & cenſus numero coequantur, a rebus* — $px + x^2 = q$.
*Dimidio ſumpto, cenſum producere debes*
*Addereque numero, cujus a radice totiens*
*Tolle ſemis rerum, cenſus latuſque redibit.*

2. *Et ſi cum rebus drachme quadrato pares ſunt,* — $q + px = x^2$.
*Adde, ſicut primo, numerum producto quadrato*
*E rebus mediis, hujuſque radice recepta,*
*Si rebus mediis addes cenſus pateſiet.*

3. *At ſi cum numero radices cenſus æquabit,* — $x^2 + q = px$.
*Drachmas a quadrato deme rei medietatis,*
*Hujus quod ſuperit radicem adde traheve*
*A rebus mediis ſic cenſus coſta noteſcet.*

Quartam formulam non admiſerunt vetuſtiores, utpote quæ ambos valores $x$ parit negativos, ut mox videbitur, negativos autem valores pro falſis habuerunt.

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Hinc quoque in omni æquatione quadraticâ affectâ, binæ exiſtunt radices ſeu valores quantitatis incognitæ, quæ evadent ſimul reales, ubi $q$ ſit affirmativa, aut ubi ſit negativa et non major ſaltem quam $\frac{1}{4}p^2$; nam ſi major, evadent ſimul impoſſibiles.

### De radicibus æquationum quadraticarum.

31. Ex radicibus ſeu valoribus quantitatis incognitæ realibus, affirmativi deſignant in genere quantitates *additiias*, negativi *ablatitias*. In quæſtionibus profecto ſolvendis quæ circa abſtractas quantitates verſantur, ubi nulla ſit rerum contrarietas, negliguntur negativi valores quaſi inutiles, et affirmativis tantum locus conceditur. Atque hinc quidem ablegabant Algebriſtæ vetuſtiores (necnon ablegant etiamnum nonnulli hujuſce ævi) radices negativas ſecundæ et tertiæ formulæ, et quartam omnino formulam. Verùm in mixta Matheſi, ubi oppoſitio quædam inter quantitates de quibus agitur, incidere poteſt, velut in motu locali inter progreſſum et regreſſum, in rebus humanis inter bona et debita, in phyſicis inter attractionem et repulſionem, jure admittendæ ſunt, utpote quæ ad caſus quæſtionum datis contrarios reſpiciunt. Negativarum naturam primus perſpexiſſe videtur acutiſſimus *Carteſius*, qui primus harum uſum in analyſin & geometriam introduxit.

32. Ex binis radicibus affirmativis tertiæ formulæ, haud raro contingit quod una tantum quæſtioni unde elicitur ſolvendæ reſpondet; nam quadratum $x^2 - px + \frac{1}{4}p^2$ ambigue pariet radix $x - \frac{1}{2}p$ vel $\frac{1}{2}p - x$, prout $x$ ſit major vel minor quam $\frac{1}{2}p$. Unde ex binis valoribus per extractionem prodeuntibus major, ſcilicet $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$ deſignabit $x$ in priore caſu, minor $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$ in poſteriore.

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Cæterum in quæſtione ad hujuſmodi æquationem perducente, ſæpe ex profeſſo vel tacite, definitur quanta ſit $x$ reſpectu $\frac{1}{2}p$.

33. Radices impoſſibiles propriè nec ſunt affirmativæ neque negativæ, licet ſub utraque forma deliteſcunt. Indicant profecto, quæſtiones unde naſcuntur ſolutionem prorſus reſpuere. Et dari quidem radices ſæpe impoſſibiles æquum eſt, ne caſus problematum, qui ſæpe impoſſibiles ſunt, exhibeant poſſibiles.

Hæc omnia de quadraticis abſtracte jam poſita, ex problematum inſequentium ſolutione luculentius patebunt.

## P R O B. I.

34. *Vir quidam nummos* a *in mercaturam impendit, quum vero hôc non ſuccederet, ſe expedire cupit ſub fine anni primi ; at occaſione non oblatâ, ſolvi nequit niſi ſub fine anni ſecundi : quo tempore, ex nummis poſt primi anni jacturam redeuntibus deperit* b. *Quæritur damnum ejus pro ſingulis* £100 *quotannis.*

Deſignante $x$ damnum quæſitum, deventum erat § 15, ad æquationem $a \times \overline{1 - \frac{x}{100}}\Big|^2 = a \times \overline{1 - \frac{x}{100}} - b$, quæ concinnando prodit $x^2 - 100x = -10000\frac{b}{a}$. Et complendo quadratum, ſecundum Regulam § 29, erit $x^2 - 100x + 2500 = 2500 - 10000\frac{b}{a} = 2500 \times \overline{1 - \frac{4b}{a}}$. et extracta radice quadratica, $x - 50 = \pm 50\sqrt{1 - \frac{4b}{a}}$. adeoque $x = 50 \pm 50\sqrt{1 - \frac{4b}{a}}$.

Exempl. 1$^{mo}$. Sit $b$ ad $a$ in ratione 1 ad $4\frac{1}{6}$ ſeu 6 ad 25; erit $\frac{b}{a} = \frac{6}{25}$ et $\sqrt{1-\frac{4b}{a}} = \sqrt{1-\frac{24}{25}} = \frac{1}{5}$. Adeoque $x = 50 \pm 50 \times \frac{1}{5} = 60$ vel 40. Ex his vero uterque quæſtionem ſolvet. Nam ſi ſumatur $x = 60$ *per cent.* poſito quod $a$ ſit £100000, ſecundum proportionem allatam debet eſſe $b =$ £24000. Cæterum *mercatoris* damnum ſub fine anni primi prodit £60000, reſiduum igitur ſit £40000: ex quo $b$ ſeu damnum ſub fine anni ſecundi, prodit £24000, ut æquum eſt. Et ſi $x = 40$ *per cent.* erit damnum ſub fine anni primi £40,000 et reſiduum £60,000: adeoque $b =$ £24,000, ut antea. 2$^{do}$. Sit $b$ ad $a$ in ratione 1 ad 4; evaneſcente jam $\pm 50 \sqrt{1-\frac{4b}{a}}$ prodit unicus valor $x$, ſcil; 50, qui quæſtioni ſolvendæ reſpondet; ut conſtat. 3$^{tio}$. Sit $b$ ad $a$ in ratione 1 ad 3; prodibit $x = 50 \pm 50 \sqrt{-\frac{1}{3}}$, uterque ſcil; valor $x$ impoſſibilis. Quod indicio eſt hunc quæſtionis caſum neutiquam eſſe ſolvendum.

## P R O B. II.

35. *Lapide in puteum decidente, ex ſono lapidis fundum percutientis, altitudinem putei cognoſcere.*

Sit altitudo putei $x$, et ſi lapis (neglectâ aeris reſiſtentiâ) motu uniformiter accelerato deſcendat per ſpatium quodlibet datum $a$ (puta 16,122 pedum,) in tempore dato $b$ (unius ſcil; minuti ſecundi,) et ſonus, (cum percurrit 1142 pedes in tempore unius minuti ſecundi motu uniformi) tranſeat per idem datum ſpatium $a$ in tempore dato $d$; lapis deſcendet per ſpatium $x$ in tempore $\frac{b\sqrt{x}}{\sqrt{a}}$, (nam ob tempora deſcenſus gravium ut radices ſpatiorum, erit hoc tempus ad tempus datum $b$ ut $\sqrt{x}$ ad $\sqrt{a}$,) ſonus autem

qui fit a lapide in fundum putei impingente aſcendet per idem ſpatium $x$ in tempore $\frac{dx}{a}$. (Erit enim hoc tempus ad tempus datum $d$, ob motum ſoni uniformem, ut ſpatium $x$ ad ſpatium $a$.) Ex horum temporum ſumma, conflatur tempus a lapide demiſſo ad ſoni reditum. Hoc tempus ex obſervatione cognoſci poteſt: ſit ipſum $t$, et erit $\frac{b\sqrt{x}}{\sqrt{a}} + \frac{dx}{a} = t$. Ac $\frac{b\sqrt{x}}{\sqrt{a}} = t - \frac{dx}{a}$. Et partibus quadratis, $\frac{b^2x}{a} = t^2 - \frac{2tdx}{a} + \frac{d^2x^2}{a^2}$. Et concinnando $x^2 = \frac{2adt+ab^2}{d^2}x - \frac{a^2t^2}{d^2}$. Et inde ope formulæ III, conferendo $\frac{2adt+ab^2}{d^2}$ cum $p$, et $\frac{a^2t^2}{d^2}$ cum $q$; evadet $\frac{1}{2}p = \frac{2adt+ab^2}{2d^2}$. $\frac{1}{4}p^2 - q = \frac{4a^2d^2t^2+4a^2b^2dt+a^2b^4}{4d^4} - \frac{a^2t^2}{d^2} = \frac{4a^2d^2t^2+4a^2b^2dt+a^2b^4-4a^2d^2t^2}{4d^4} = \frac{4a^2b^2dt+a^2b^4}{4d^4} = \frac{a^2b^2}{4d^4} \times \overline{b^2+4dt}$. Et $\sqrt{\frac{1}{4}p^2-q} = \frac{ab}{2d^2}\sqrt{b^2+4dt}$. Adeoque $x = \frac{adt+\frac{1}{2}ab^2}{d^2} \pm \frac{ab}{2d^2}\sqrt{b^2+4dt}$. Cæterum altitudo putei $x$ duplex eſſe nequit, hanc igitur exponet minor radix ſeu valor $x$, ſcilicet $\frac{adt+\frac{1}{2}ab^2}{d^2} - \frac{ab}{2d^2}\sqrt{b^2+4dt}$. Nam tempora $\frac{b\sqrt{x}}{\sqrt{a}}$ et $\frac{dx}{a}$ ex hypotheſi ſimul conficiunt $t$ tempus obſervatione compertum, erit igitur tempus $\frac{dx}{a}$ ſeorſim minus quam $t$, ſive $x$ minor quam $\frac{at}{d}$; quod in hoc caſu ita ſe habet. Liquet enim $\sqrt{b^2+4dt}$ eſſe majorem quam $b$; finge æqualem $b+c$, eritque $x = \frac{adt+\frac{1}{2}ab^2}{d^2} - \frac{\frac{1}{2}ab}{d^2}\overline{b+c} = \frac{adt+\frac{1}{2}ab^2-\frac{1}{2}ab^2-\frac{1}{2}abc}{d^2} = \frac{adt-\frac{1}{2}abc}{d^2}$ et proinde $x$ minor quam $\frac{adt}{d^2} = \frac{at}{d}$.

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36. Exempl. Sit tempus obſervatione compertum a lapide demiſſo ad ſoni reditum, quinque minutorum ſecundorum, ſeu $t = 5''$. Adeoque, cum $a =$ ped. 16,122. $b = 1''$. et $d$ propter analogiam $= ,014117''$. (nam ped. 1142 : ped. 16,122 :: $1''$ : $,014117''$) evadet $adt = 16,122 \times 5 \times ,014117 = 1,13797137$. $\frac{1}{2}ab^2 = 8,061$. et $\frac{adt + \frac{1}{2}ab^2}{d^2} = 46158,79$. Erit quoque $b^2 + 4dt = 1,28234$. $\sqrt{b^2 + 4dt} = 1,1324$. $\frac{ab}{2d^2} = 40448,6556$. $\frac{ab}{2d^2}\sqrt{b^2+4dt} = 45804,057$

Et proinde, $\frac{adt + \frac{1}{2}ab^2}{d^2} - \frac{ab}{2d^2}\sqrt{b^2+4dt} = 354,733$ ped.

Altitudinem putei recte ſtatui ped. 354,733. ſic probare licet. Tempora $b\sqrt{\frac{x}{a}}$ et $\frac{dx}{a}$ per hypotheſin ſimul æquari debent $t$ ſive quinque minutis ſecundis. Sed $x = 354,733$. $\frac{x}{a} = 22,0030$. et $\sqrt{\frac{x}{a}}$ id eſt, $b\sqrt{\frac{x}{a}} = 4,69''$. Eſt quoque $dx = 5,00776$. et $\frac{dx}{a} = ,31''$. Et proinde $b\sqrt{\frac{x}{a}} + \frac{dx}{a} = 4,69''. + ,31''. = 5''$.

37. Alter valor $x$, ſcilicet $\frac{adt + \frac{1}{2}ab^2}{d^2} + \frac{ab}{2d^2}\sqrt{b^2+4dt}$ alio ſatisfacit problematis caſui; cujus ea eſt conditio ut tempus obſervatione compertum ſit exceſſus temporis quo aſcendit ſonus ſupra tempus quo decidit lapis, ſeu $\frac{dx}{a} - b\sqrt{\frac{x}{a}} = t$. nam hæc æquatio, tranſponendo et quadrando dabit $\frac{b^2x}{a} = t^2 - \frac{2dtx}{a} + \frac{d^2x^2}{a^2}$ ut antea. Adeoque per reductionem $x = \frac{adt + \frac{1}{2}ab^2}{d^2} \pm \frac{ab}{2d^2}\sqrt{b^2+4dt}$.

38. In problematum phyſicorum reſolutione obſervandum eſt, quod concluſionum algebraicè elicitarum auctoritas, pendet ex principiorum phyſicorum quibus innituntur certitudine et accuratione, tum ex ipſorum experimentorum fide. Sic in hac ſolutione minus accurata prodit concluſio, 1^mo^. ex eo quod negligitur aeris reſiſtentia, quâ gravia cadendo notabiliter retardantur; 2^do^. quod valores $a$ et $d$ vix forſan accurate determinandi ſunt; et 3^tio^. quod in ipſum $t$ parvus error, unius ſemiſecundi ſaltem, irrepere poteſt; verùm hoc errorem haud contemnendum in concluſionem inducet.

## P R O B. III.

39. *Invenire punctum æqualis attractionis inter duos planetas ſitum, quorum innoteſcit diſtantia ab invicem, et ratio maſſarum.*

Sit $d$ diſtantia lunæ a tellure, $x$ diſtantia puncti quæſiti a luna, erit hujus diſtantia a tellure $d-x$. Et quoniam harum attractiones quibuſcunque diſtantiis ſunt ut maſſæ $t$ et $l$ directe, et quadrata diſtantiarum inverſe; erit attractio telluris ut $\frac{t}{(d-x)^2}$, lunæ autem ut $\frac{l}{x^2}$. Adeoque hìc poſita attractionum æqualitate, erit $\frac{t}{(d-x)^2}=\frac{l}{x^2}$. et concinnando, $x^2+\frac{2dl}{t-l}x=\frac{d^2l}{t-l}$. Et collatis hujus terminis cum formula I. erit $\frac{1}{2}p=\frac{dl}{t-l}$. $\frac{1}{4}p^2+q=\frac{d^2l^2}{(t-l)^2}+\frac{d^2l}{t-l}$ $=\frac{d^2lt}{(t-l)^2}=\frac{d^2}{(t-l)^2}\times lt$. Et $\sqrt{\frac{1}{4}p^2+q}=\frac{d}{t-l}\sqrt{lt}$. Adeoque $x=\frac{-dl\pm d\sqrt{lt}}{t-l}$.



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Horum valorum $x$, affirmativus $\frac{-dl+d\sqrt{lt}}{t-l}$ quæſtionem ſolvet: negativus vero $\frac{-dl-d\sqrt{lt}}{t-l}$ ad alterum quoddam punctum æqualis attractionis ultra lunam ſitum reſpicit; ut ex eo conſtat, quod ſi quæratur alterum punctum iſtud, diſtantiis jam deſignatis per $x$ et $d+x$, habebitur æquatio $\frac{t}{(d+x)^2}=\frac{l}{x^2}$. Et per reductionem, $x=\frac{dl\pm d\sqrt{lt}}{t-l}$. idem ſcilicet geminus valor $x$ ut antea, at contrariis ſignis affectus: quorum affirmativus hunc caſum ſolvet, negativus ad priorem refertur.

40. Et quidem binos valores $x$ in utroque caſu diverſa ſortiri ſigna æquum eſt, quum duo hæc puncta æqualis attractionis ad plagas *lunæ oppoſitas* ſiſtantur. Verùm poſito quod $x$ ſit diſtantia puncti quæſiti a tellure, erit $d-x$ diſtantia ejuſdem a luna. Adeoque $\frac{t}{x^2}=\frac{l}{(d-x)^2}$. unde per reductionem, prodit $x=\frac{dt\pm d\sqrt{lt}}{t-l}$. uterque jam valor $x$ affirmativus, ut oportet; quum ad *eandem* plagam, reſpectu *telluris*, bina attractionis æqualis puncta ſiſtantur. Hinc liquet, quod quantitates affirmativæ et negativæ, conceptibus noſtris tantum diſcrepantes, in rerum naturâ æque reales ſunt habendæ. Ex his autem valoribus, minor $\frac{dt-d\sqrt{lt}}{t-l}$ quæſtionem ſolvet. Sit enim $d=60$ ſemidiametris telluris, et $t$ ad $l$ in ratione 40 ad 1 circiter; evadet diſtantia puncti quæſiti a tellure, $=\frac{2400-379{,}464}{39}=51{,}8$ ſeu 52 ſemidiam. telluris fere, et diſtantia alterius puncti prodibit $\frac{2400+379{,}464}{39}=72$ ſemidiam. fere.

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41. Reductio æquationis $\frac{t}{x^2} = \frac{l}{(d-x)^2}$ aliter inſtitui poteſt. Nam extracta radice quadratica erit $\frac{\sqrt{t}}{x} = \frac{\pm\sqrt{l}}{d-x}$. Et concinnando, $x = \frac{\pm d\sqrt{t}}{\sqrt{t}+\sqrt{l}}$. ſeu multiplicando numeratorem et denominatorem per $\sqrt{t}-\sqrt{l}$, prodit $x$ ut antea $= \frac{dt \pm d\sqrt{lt}}{t-l}$.

## P R O B. IV.

42. *Rectam datam in extrema et media proportione ſecare.*

Sit rectæ datæ $a$ ſegmentum majus $x$, erit ſegmentum minus $a-x$, adeoque per quæſtionem, $x^2 = a \times \overline{a-x} = -ax + a^2$. Et per reductionem, $x = -\frac{1}{2}a + \sqrt{\frac{5}{4}a^2}$. Alter valor $x$ qui prodit negativus, ſcil; $-\frac{1}{2}a - \sqrt{\frac{5}{4}a^2}$ diverſo problematis caſui reſpondet, nempe, *Rectam datam in extrema et media proportione producere.* Hoc eſt, ita ut quadratum augmenti $x$ ſit æquale rectangulo ſub $a$ et $a+x$ contento, ſive $x^2 = ax + a^2$. Quæ per reductionem dabit eoſdem valores $x$, at contrariis ſignis affectos: ex his autem, $\frac{1}{2}a + \sqrt{\frac{5}{4}a^2}$ hunc quæſtionis caſum ſolvet.

## P R O B. V.

43. *Mercatores* A *et* B *pro mercibus* per annum *venumdatis, eandam ſummam* £2112 *recipiunt; verùm* A *lucrando* 2 per cent. *plus quam* B, *nummos ſuos adauget incremento* £100 per annum *plus quam* B. *Quæritur utriuſque lucrum* per cent. *et totum utriuſque lucrum* per annum.

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Sit $x$ lucrum *per cent.* mercatoris *A*, eritque propter analogiam, (ut $100+x$, reditus pro £100 impenſis, ad $x$; ita 2112, reditus totus *per annum* ad) $\frac{2112x}{100+x}$ totum lucrum mercatoris *A per annum*. Et ſimiliter, $(98+x : x-2 :: 2112 :)$ $\frac{2112x-4224}{98+x}$ totum lucrum *B per annum*. Adeoque per quæſtionem, $\frac{2112x}{100+x} - \frac{2112x-4224}{98+x} = 100$. Et concinnando, $x^2 = -198x - 5576$. Et per reductionem, $x = -99 \pm 65$. Prodeunte igitur utroque valore negativo, colligere licet $x$ in hoc caſu deſignare non lucrum ſed damnum, et quæſtionem aliter eſſe efferendam. Scilicet,

44. *Mercatores* A *et* B *pro mercibus, &c. eandem ſummam £*2112 *recipiunt. Verùm* A, *amittendo* 2 per cent. *minus quam* B, *amittit* £100 per annum *minus quam* B. *Quæritur utriuſque damnum* per cent. *et totum utriuſque damnum* per annum.

Deſignante jam $x$ damnum *per cent.* mercatoris *A*, erit $(100-x : x :: 2112 :)$ $\frac{2112x}{100-x}$ totum damnum *A per annum*. Et ſimiliter, $(98-x : x+2 :: 2112 :)$ $\frac{2112x+4224}{98-x}$ totum damnum *B per annum*. Sed per quæſtionem, $\frac{2112x+4224}{98-x} - \frac{2112x}{100-x} = 100$. Et concinnando, $x^2 = 198x - 5576$. Et per reductionem, $x = 99 \pm 65$. Horum valorum $x$ affirmative jam prodeuntium, minor ſcil; $99-65=34$, quæſtionem ſolvet. Erit enim damnum *A*, £34 *per cent.* et totum damnum ejus *per annum* $\frac{2112 \times 34}{66} =$ £1088. Eritque damnum *B*, £36 *per cent.* et totum ejus damnum *per annum* $\frac{2112 \times 34 + 4224}{64} =$ £1188.

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Alter valor $x$, ſcil; $99 + 65 = 164$, utpote major quam 100, damnum pro ſingulis £100 impenſis deſignare nequit, verùm caſui inſequenti ſolvendo reſpondet.

45. *Mercatorum* A *et* B *lucrum* per annum *nummos ab utroque in mercaturam impenſos excedit ſummâ* £2112. *At* A *quum lucratus ſit* 2 per cent. *minus quam* B, *lucratur* £100 per annum *plus quam* B. *Quæritur utriuſque lucrum* per cent. *et totum utriuſque lucrum* per annum.

Deſignante $x$ lucrum *per cent.* mercatoris $A$, erit, $(x - 100 : x :: 2112 :) \frac{2112x}{x-100}$ totum lucrum $A$ *per annum.*

Et ſimiliter, $x - 98 : x + 2 :: 2112 :) \frac{2112x + 4224}{x - 98}$ totum lucrum $B$ *per annum.* Sed per quæſtionem, $\frac{2112x}{x-100} - \frac{2112x + 4224}{x-98} = 100$. Et concinnando, $x^2 = 198x - 5576$, ut antea. Adeoque per reductionem, $x = 164$. Erit igitur lucrum $A$ *per cent.* £164, et totum lucrum ejus *per annum* $\frac{2112 \times 164}{64} =$ £5412. Et lucrum $B$ *per cent.* £166, et totum lucrum ejus *per annum* $\frac{2112 \times 164 + 4224}{66} =$ £5312.

## P R O B. VI.

46. *Numerum* 10 *ita in duas partes dividere, ut productum ex partibus in ſe ductis conficiat* 30.

Sit $x$ una pars, alteram deſignabit $10 - x$, eritque $x \times \overline{10 - x} = 30$, et per reductionem, $x = 5 \pm \sqrt{-5}$. U-

terque igitur prodit valor $x$ impoſſibilis: et jure quidem, nam conditio ut factum ſub partibus conficiat 30, eſt impoſſibilis, cum factum maximum (ubi ſcilicet partes inter ſe æquantur) ſit 25.

Deſignante enim in genere $d$ dimidium quantitatis datæ, ſit una pars $d+y$, alteram deſignabit $d-y$. Sed harum productum $\overline{d+y}\times\overline{d-y}=d^2-y^2$ fit maximum, ut conſtat, ubi evaneſcente $y^2$ deſignatur tantum per $d^2$. Quod ex *Euclid. lib.* 2, *prop.* 5, quoque conſtat.

47. Hoc problema (et ſimilia) verſione in ſermonem Algebraicum paulum immutatâ, ad puram quadraticam facile redigere licet. Sit enim pars major $5+y$, minorem deſignabit $5-y$. Adeoque erit $\overline{5+y}\times\overline{5-y}=30$. Sive, $25-y^2=30$. Adeoque $y^2=-5$. Seu $y=\pm\sqrt{-5}$ uterque valor impoſſibilis ut antea.

## De reductione æquationum quæ ordini quadraticarum adscribendæ sunt.

48. Æquationes parium dimenſionum formæ $x^{2n}=.px^n.q.$ (ubi duos tantum terminos ingreditur incognita quantitas, et index termini altiſſimi ſit duplus ſecundi) juxta legem quadraticarum, ad puras dimenſionis dimidiatæ ſunt deprimendæ; quas poſtea, ſi index $n$ ſit integer per evolutionem, ſi fractus per involutionem reſolvere licet. Nam $x^{2n}=.px^n.q$ extracta radice quadratica, prodit $x^n=.\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2.q}$ adeoque, $x=\sqrt[n]{.\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2.q}}$. Problemata ſequentia hujuſmodi æquationes pariunt.

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## P R O B. I.

49. *Invenire duos numeros quorum productum ſit* 100, *et differentia radicum quadraticarum* 3.

Sit minor $x$, majorem deſignabit $\frac{100}{x}$. Adeoque erit $\frac{10}{\sqrt{x}} - \sqrt{x} = 3$. ſive concinnando, $x = -3\sqrt{x} + 10$. et per reductionem, $\sqrt{x} = \frac{-3 \pm 7}{2} = 2$ vel $-5$. Et per involutionem, $x = 4$ vel $25$. Erit itaque $x$ in hoc caſu 4, et alter numerus 25: aut vice verſa, ſi ſtatuatur $x$ major numerus.

## P R O B. II.

50. *Invenire duos numeros quorum productum ſit* 1, *et ſumma quadratorum* 6.

Deſignante $x$ unum numerum, alterum deſignabit $\frac{1}{x}$. Adeoque erit $x^2 + \frac{1}{x^2} = 6$. Seu concinnando, $x^4 = 6x^2 - 1$. Et per reductionem, $x^2 = 3 \pm \sqrt{8}$. Et radice extractâ, $x = \pm\sqrt{3 \pm \sqrt{8}}$. Quadruplex nimirum prodit valor $x$, ſcil. $+\sqrt{3+\sqrt{8}}$, $+\sqrt{3-\sqrt{8}}$, $-\sqrt{3+\sqrt{8}}$, vel $-\sqrt{3-\sqrt{8}}$ ex quibus duo poſteriores, cum ſint iidem ac priores ſed contrariis ſignis affecti, utplurimum negliguntur. Quod in ſequentibus quoque obſervandum. Ex valoribus autem

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receptis uterque ſolvet problema. Nam $\sqrt{3+\sqrt{8}} = \sqrt{5,828}$ $= 2,414$. Adeoque, erit $x^2 = 5,828$. $\frac{1}{x^2} = 0,171$. Et horum ſumma prodit 5,999, ſeu 6 quamproxime. Sic quoque alter valor $x$, $\sqrt{3-\sqrt{8}}$ problema ſolvet quamproxime.

## PROB. III.

51. *Invenire duos numeros, quorum quadrata ſimul conficiant* $8\sqrt{2}$, *et quadratorum productum ſit* 8.

Sit $x^2$ quadratum unius numeri, erit $8\sqrt{2} - x^2$ quadratum alterius, adeoque $x^2 \times \overline{8\sqrt{2} - x^2} = 8$. Seu concinnando, $x^4 = 8\sqrt{2}\,x^2 - 8$. Et per reductionem, $x^2 = \sqrt{32} \mp \sqrt{24}$. Et radice iterum extracta, $x = \sqrt{\sqrt{32} \pm \sqrt{24}}$.

Haud ſecus in ſpeciebus, $x^4 = (2gh + 4f^2)\,x^2 - g^2h^2$ per reductionem, prodit $x^2 = gh + 2f^2 \pm \sqrt{4f^2gh + 4f^4}$. Et extracta radice, $x = \sqrt{gh + 2f^2 \pm 2f\sqrt{gh + f^2}}$.

Aliquando radices quadraticæ hujuſmodi quantitatum ex radicalibus aut ex integris et radicalibus ad invicem incommenſurabilibus compoſitarum, accurate deſignari poſſunt per ſimpliciores quantitates radicales; ope regulæ inſequentis, quam ex *Bombellio* expreſſit *Newtonus*.

52. *Deſignet* A *quantitatis alicujus partem majorem*, B *partem minorem: et erit* $\frac{A+\sqrt{A^2-B^2}}{2}$ *quadratum majoris partis radicis; &* $\frac{A-\sqrt{A^2-B^2}}{2}$ *quadratum partis minoris, quæ quidem majori adnectenda eſt cum ſigno ipſius* B. *Adeoque* $\sqrt{\frac{A+\sqrt{A^2-B^2}}{2}}$ $\pm \sqrt{\frac{A-\sqrt{A^2-B^2}}{2}}$ *radix quæſita.*

Sit enim $y \pm 2\sqrt{yz} + z$ quantitas aliqua cujus radix $\sqrt{y} \pm \sqrt{z}$ extrahenda eſt. Deſignet $A$ partem ejus *majorem* $y+z$, et $B$ partem *minorem* $2\sqrt{yz}$ [d], quo $\sqrt{A^2-B^2}$ ſit poſſibilis. Quadrando, erit $y^2+2yz+z^2=A^2$, et $4yz=B^2$. Et ſubducendo poſteriorem, $y^2-2yz+z^2=A^2-B^2$. Et extracta radice, $y-z=\sqrt{A^2-B^2}$. Adeoque, $\overline{y+z}+\overline{y-z}$ ſeu $2y=A+\sqrt{A^2-B^2}$ et $y=\frac{A+\sqrt{A^2-B^2}}{2}$. Itemque $\overline{y+z}-\overline{y-z}$ ſeu $2z=A-\sqrt{A^2-B^2}$ et $z=\frac{A-\sqrt{A^2-B^2}}{2}$. Adeoque $\sqrt{y}\pm\sqrt{z}=\sqrt{\frac{A+\sqrt{A^2-B^2}}{2}}\pm\sqrt{\frac{A-\sqrt{A^2-B^2}}{2}}$.

Ut ſi quantitatis $3\pm\sqrt{8}$ radix extrahenda eſt, ſcribe 3 pro $A$ et $\sqrt{8}$ pro $B$, eritque $\sqrt{A^2-B^2}=1$. Et quadrata partium radicis ſimul, ſcil. $\frac{A+\sqrt{A^2-B^2}}{2}\pm\frac{A-\sqrt{A^2-B^2}}{2}=\frac{3+1}{2}\pm\frac{3-1}{2}=2\pm1$. Ergo radix eſt $1\pm\sqrt{2}$. Hæc enim quadrando dabit $1+2\pm2\sqrt{2}=3\pm\sqrt{8}$. Eodem modo ſi de $2f^2+gh\pm2f\sqrt{f^2+gh}$ radix extrahi debet, ſcribendo $2f^2+gh$ pro $A$, et $2f\sqrt{f^2+gh}$ pro $B$, erit $\sqrt{A^2-B^2}=gh$, adeoque $\frac{A+\sqrt{A^2-B^2}}{2}=f^2+gh$ et $\frac{A-\sqrt{A^2-B^2}}{2}=f^2$. Ergo radix ipſa $f\pm\sqrt{f^2+gh}$. Haud ſecus, ſi de quantitate impoſſibili $1\pm4\sqrt{-3}$ radix ſit extrahenda, poſito $A=1$, $B=4\sqrt{-3}$, et $B^2=-48$, erit $\sqrt{A^2-B^2}=\sqrt{1+48}=7$.

[d] Sit quantitatis alicujus dimidium $d$, et pars major $d+x$, erit pars minor $d-x$. Sed rectangulum ſub partibus $\overline{d+x}\times\overline{d-x}=d^2-x^2$. et bis rectangulum $2d^2-2x^2$, ſit *minus* quam $\overline{d+x}^2+\overline{d-x}^2=2d^2+2x^2$ quadrata partium ſimul. Quod ex *Euclid. Lib.* 2, *prop.* 4 *et* 5 quoque colligere licet.

Et quadrata partium radicis, $= \frac{1+7}{2} \pm \frac{1-7}{2} = 4 + -3$. Ergo, radix ipſa $2 \pm \sqrt{-3}$. Rurſus, ſi habeatur $2\sqrt{-1}$, poſito $A = 0$. $B = 2\sqrt{-1}$, $B^2 = -4$, erit $\sqrt{A^2 - B^2} = \sqrt{0+4} = 0 + 2$. Adeoque radix $= \sqrt{\frac{0+0+2}{2}} + \sqrt{\frac{0-0-2}{2}} = 1 + \sqrt{-1}$. Porro, ſi ex $\sqrt{32} \pm \sqrt{24}$, ex radicalibus ſolis conflatâ, radix extrahenda ſit; ponendo $\sqrt{32}$ pro $A$, et $\sqrt{24}$ pro $B$, erit $\sqrt{A^2 - B^2} = \sqrt{8}$, et inde quadratum majoris partis radicis $\frac{\sqrt{32} + \sqrt{8}}{2} = \frac{2\sqrt{8} + \sqrt{8}}{2} = \frac{3\sqrt{8}}{2} = 3\sqrt{2} = \sqrt{18}$. Et quadratum minoris, $\frac{\sqrt{32} - \sqrt{8}}{2} = \sqrt{2}$. Radix itaque eſt $\sqrt[4]{18} \pm \sqrt[4]{2}$. Denique, ſi habeatur $6 + \sqrt{8} - \sqrt{12} - \sqrt{24}$, ponendo $6 + \sqrt{8} = A$ et $-\sqrt{12} - \sqrt{24} = B$, fiet $A^2 = 44 + 12\sqrt{8}$, $B^2 = 36 + 2\sqrt{288} = 36 + 12\sqrt{8}$, et $\sqrt{A^2 - B^2} = \sqrt{8}$, unde quadratum majoris partis radicis erit $\frac{6 + 2\sqrt{8}}{2} = 3 + \sqrt{8}$, atque illud minoris $= 3$, adeoque radicis pars major $\sqrt{3 + \sqrt{8}}$, hoc eſt, ut ſupra $1 + \sqrt{2}$; et pars minor $\sqrt{3}$. Et proinde radix ipſa $1 + \sqrt{2} - \sqrt{3}$.

53. Cæterum ubi plures ſunt hujuſmodi termini radicales ad invicem incommenſurabiles, excogitavit ſagaciſſimus *Newtonus* methodum citius inveniendi partes radicis; *dividendo ſcilicet factum quarumvis duarum radicalium per tertiam aliquam radicalem quæ producit quotum rationalem et integrum, donec tot eruantur hujuſmodi quoti diverſi quot ſunt partes radicis quæſitæ. Nam dimidii quoti cujuſque radix erit pars radicis.*

Nam ſi tres habeantur hujuſmodi radicales, ſit radix quæſita, $\sqrt{x} \pm \sqrt{y} \pm \sqrt{z}$, quadrando erit $x + y + z \pm 2$

$\sqrt{xy} \pm 2\sqrt{xz} \pm 2\sqrt{yz} = A \pm B$. At trium rectangulorum unoquoque duas radicis partes continente, ſi dividatur factum duorum quorumvis, tres utique radicis partes comprehendentium, per tertium, duas itidem radicis partes continens, quotiens unam radicis partem involvet: ſcil. $\frac{2\sqrt{xy} \times 2\sqrt{xz}}{2\sqrt{yz}} = 2x$, $\frac{2\sqrt{xy} \times 2\sqrt{yz}}{2\sqrt{xz}} = 2y$, et $\frac{2\sqrt{xz} \times 2\sqrt{yz}}{2\sqrt{xy}} = 2z$. Ergo radix dimidii cujuſque quoti erit pars radicis quæſitæ. Sic, in exemplo noviſſimo $\frac{\sqrt{8} \times \sqrt{12}}{\sqrt{24}} = 2$, $\frac{\sqrt{8} \times \sqrt{24}}{\sqrt{12}} = 4$, $\frac{\sqrt{12} \times \sqrt{24}}{\sqrt{8}} = 6$. Ergo partes radicis ſunt $1$, $\sqrt{2}$, $\sqrt{3}$. ut ſupra.

Ad ſigna partium quod attinet, quoniam factum partium duarum ſignis ſimilibus utentium erit affirmativum, ſecus vero negativum: erunt partes radicis $1$ et $\sqrt{2}$ quas continet prima radicalis $+\sqrt{8}$ $(= 2\sqrt{xy})$ ambæ affirmativæ, vel ambæ negativæ; ſecundæ autem radicalis $-\sqrt{12}$ $(= -2\sqrt{xz})$ partes, ſcil. $1$ et $\sqrt{3}$ contrariis affici debent ſignis. Radix igitur erit vel $1 + \sqrt{2} - \sqrt{3}$, vel $-1 - \sqrt{2} + \sqrt{3}$.

54. Si quatuor ſint hujuſmodi radicales, ſit radix $\sqrt{v} \pm \sqrt{x} + \sqrt{y} \pm \sqrt{z}$. quadrando erit $v + x + y + z \pm 2\sqrt{vx} \pm 2\sqrt{vy} \pm 2\sqrt{vz}$; $\pm 2\sqrt{xy} \pm 2\sqrt{xz}$; $\pm 2\sqrt{yz} = A \pm B$. ex 15 autem diverſis horum rectangulorum combinationibus quas inſtituere licet, tripliciter erui poteſt quæque radicis pars, unde duodecim prodibunt quoti rationales et integri: ſcil. $\frac{2\sqrt{vx} \times 2\sqrt{vy}}{2\sqrt{xy}}$, vel $\frac{2\sqrt{vx} \times 2\sqrt{vz}}{2\sqrt{xz}}$, vel $\frac{2\sqrt{vy} \times 2\sqrt{vz}}{2\sqrt{yz}} = 2v$; $\frac{2\sqrt{vy} \times 2\sqrt{xy}}{2\sqrt{vx}}$ vel &c. $= 2x$; et ſic deinceps. Tres autem qui ſuperſunt quoti evadent irrationales, quum facta jam reſtantia, ſcil. $2\sqrt{vx} \times 2\sqrt{yz}$, $2\sqrt{vy} \times 2\sqrt{xz}$, et $2\sqrt{vz} \times 2\sqrt{xy}$,

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utpote quæ omnes radicis partes continent; diviſorem inter reliqua rectangula nullum ſortientur qui quotum rationalem & integrum pariet.

55. Ex dictis colligere licet, quod radices hujuſmodi quantitatum vel ex integris et radicalibus, vel ex radicalibus ſolis compoſitarum ope regulæ præfatæ quidem extrahere fas erit, ubi $\sqrt{A^2-B^2}$ exhibeat quantitatem rationalem et integram, vel ſi non, ubi radicales quas exhibent A et $\sqrt{A^2-B^2}$ extrahendo quicquid eſt rationale, ex parte radicali conveniant.

Huic parti, *Appendicem* ſubjungam de celeberrimo *Theoremate Binomio*, cujus uſus tam in involutione et evolutione, quam in omni fere Analyſeos parte egregius.

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## De Theoremate Binomio.

56. Præclarum hoc theorema, quod tentando & inductione extudit *Newtonus*, non niſi anno 1676 epiſtolis ad *Oldenburg* miſſis, cum *Leibnitio* communicandis, patefecit; licet hoc unà cum præcipuis ſuis inventis analyticis, ſaltem ante peſtem *Londinenſem* anno 1665 ingruentem, ei innotuiſſe ex iſtis conſtat. Suſpicabatur nimirum vir egregius, haud minus modeſtiâ quam ingenio inſignis, inventa hæcce ſua, aut aliis perſpecta jam eſſe, aut ſaltem fore, prius quam ipſe ætatem ſcribendo parem attigerit [e]. Natus eſt *Newtonus* anno 1642.

57. Theorema Binomium ita profertur,

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} \times \left(1 + \frac{m}{n}Q + \frac{m}{n}.\frac{m-n}{2n}Q^2 + \frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}Q^3 \&c.\right)$$

Sive, ſecundum formulam *Newtonianam*,

$$= P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ \ \&c.$$

[e] —— "About that time the plague breaking out, obliged me to get "hence [from *Cambridge*] and think of ſomething elſe. Notwithſtanding, I "added to my other diſcoveries at that time, a certain conſtruction of Loga-"rithms deduced from the Area of the Hyperbola. —— I am aſhamed to "tell to how many places of figures I carried theſe computations, having no "other buſineſs at that time; for I pleaſed myſelf too much with theſe diſco-"veries. But as ſoon as that ingenious performance of *Nicolaus Mercator* ap-"peared [in the year 1668] I mean his *Logarithmotechnia*, (who, I doubt not, "firſt diſcovered the things he publiſhed,) I began to cool a little upon thoſe "things, ſuſpecting that either he underſtood the extraction of roots as well as "diviſion of fractions; or at leaſt, that others, upon the diſcovery of diviſion, "would find out the reſt, before I could be of a ſufficient age for writing." See his Letters to *Oldenburg*, *Commerc. Epiſtol.* N° 48 and 55; or *Stewart's Quadrature of Curves*, &c. p. 357.

Ubi $P + PQ$ deſignat quantitatem quamlibet compoſitam, ſive binomiam ſive polynomiam, cujus radix quævis, dimenſio quævis, vel radix dimenſionis inveſtiganda eſt; $P$, primum ejus terminum; $Q$, quotientem reliquorum terminorum per primum diviſorum, et proinde $PQ$, ſummam reliquorum ſive affirmativè ſive negativè ponendam; $\frac{m}{n}$, numeralem indicem radicis vel dimenſionis $P + PQ$, ſive integer ſit, ſive fractus, ſive affirmativus ſive negativus: et in formula poſteriore, A, B, C, &c. terminos inter operandum inventos; nempe A, primum terminum $P^{\frac{m}{n}}$; B, ſecundum $\frac{m}{n}AQ$; C, tertium $\frac{m-n}{2n}BQ$, &c. Porro, eodem redeunt hæ formulæ, uti conferendo inter ſe terminos utriuſque homologos manifeſtum eſt. Nam $P^{\frac{m}{n}} = P^{\frac{m}{n}} \times 1$; $\frac{m}{n}AQ = P^{\frac{m}{n}} \times \frac{m}{n}Q$; et ſic deinceps. Utramque vero poſuimus, ex eo quod prior legem progreſſionis planius exhibendo, terminis ſeriei continuandis, poſterior hiſce addendis aptior, ubi pro A, B, C, &c. ſubſtituuntur valores numerales quos ſortiuntur reſpective.

58. Theorematis Binomii complures dantur aut dari ponuntur demonſtrationes. Miſſis autem demonſtrationibus ex fluxionali computo hauſtis, ab opere elementario nimirum ablegandis; vel ex permutationum ſeu combinationum doctrinâ, non niſi prolixe elicitis: inveſtigationem ejus breviter inſtituere libet, ex principiis algebræ propriis, ad mentem *Johannis Landen*, *Treatiſe on the Reſidual Analyſis*. Cæterum neceſſe eſt ut præmittatur Theorema, Lemmatis inſtar, a *Landeno* abſque demonſtratione poſitum.

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## LEMMA.

Deſignantibus $m$ et $n$, numeros quoſlibet integros, erit,

$$\frac{p^{\frac{m}{n}}-q^{\frac{m}{n}}}{p-q}=p^{\frac{m}{n}-1}\times\frac{1+\frac{q}{p}+\overline{\frac{q}{p}}\Big|^{2}+\overline{\frac{q}{p}}\Big|^{3}\ \&c.\ldots m}{1+\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{2m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{3m}{n}}\ \&c.\ldots\ldots n}$$

Nam reſolvendo in factores ſuos, fiet,

$$\frac{p^{\frac{m}{n}}-q^{\frac{m}{n}}}{p-q}=\frac{p^{\frac{m}{n}}}{p}\times\frac{1-\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}}{1-\frac{q}{p}}$$

$$=p^{\frac{m}{n}-1}\times\frac{1-\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}}{1-\frac{q}{p}}$$ ſive, multiplicando numeratorem et denominatorem eadem quantitate $1-\overline{\frac{q}{p}}\Big|^{m}$, cujus index $m$ indicis utriuſque $\frac{m}{n}$ et 1, debet eſſe multiplus,

$$=p^{\frac{m}{n}-1}\times\frac{1-\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}}{1-\frac{q}{p}}\times\frac{1-\overline{\frac{q}{p}}\Big|^{m}}{1-\overline{\frac{q}{p}}\Big|^{m}}$$ ſive,

$$=p^{\frac{m}{n}-1}\times\frac{1-\overline{\frac{q}{p}}\Big|^{m}}{1-\frac{q}{p}}\times\frac{1-\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}}{1-\overline{\frac{q}{p}}\Big|^{m}}$$

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At ſi $\overline{1-\frac{q}{p}}\Big|^{m}$ dividatur per $1-\frac{q}{p}$ et $\overline{1-\frac{q}{p}}\Big|^{\frac{m}{n}}$ reſpective;

Fiet,

$$\frac{\overline{1-\frac{q}{p}}\Big|^{m}}{1-\frac{q}{p}} = \frac{1+\frac{q}{p}+\overline{\frac{q}{p}}\Big|^{2}+\overline{\frac{q}{p}}\Big|^{3}\ \&c.\,.\,\overline{\frac{q}{p}}\Big|^{m-1}}{1},$$ uſque ad $m$ terminos.

Itemque,

$$\frac{\overline{1-\frac{q}{p}}\Big|^{\frac{m}{n}}}{\overline{1-\frac{q}{p}}\Big|^{m}} = \frac{1}{1+\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{2m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{3m}{n}}\ \&c.\,.\,\overline{\frac{q}{p}}\Big|^{\overline{n-1}\times\frac{m}{n}}}.$$ uſque ad $n$ terminos.

Adeoque,

$$\frac{\overline{1-\frac{q}{p}}\Big|^{m}}{1-\frac{q}{p}} \times \frac{\overline{1-\frac{q}{p}}\Big|^{\frac{m}{n}}}{\overline{1-\frac{q}{p}}\Big|^{m}} = \frac{1+\frac{q}{p}+\overline{\frac{q}{p}}\Big|^{2}+\overline{\frac{q}{p}}\Big|^{3}\ \ldots m}{1+\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{2m}{n}}+\overline{\frac{q}{p}}\Big|^{\frac{3m}{n}}\ \ldots\ n}.$$

Ergo,

$$\frac{p^{\frac{m}{n}}-q^{\frac{m}{n}}}{p-q} = p^{\frac{m}{n}-1} \times \frac{1+\frac{q}{p}\ \&c.\ \ldots\ m}{1+\overline{\frac{q}{p}}\Big|^{\frac{m}{n}}\ \&c.\ \ldots\ n}.$$

Hoc Theoremate demonſtrato, ipſius Theorematis Binomii inveſtigationem expedire jam licet.

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## THEOREMATIS BINOMII INVESTIGATIO.

59. Quoniam $(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} \times (1+Q)^{\frac{m}{n}}$. inveſtiganda eſt ſeries quantitati $(1+Q)^{\frac{m}{n}}$ æqualis. Aſſumatur igitur huic æquipollere ſeriem $1 + aQ + bQ^2 + cQ^3 + dQ^4$, &c. Nec obſtat quo minus per hujuſmodi ſeriem, terminis nimirum integris indicibus perpetuo creſcentibus, rite deſignetur $(1+Q)^{\frac{m}{n}}$, cum termini ſucceſſivi intra juſtum valorem coerceri poſſint, ſi modo coefficientes $a$, $b$, $c$, ſortiantur fractos. Compendii gratiâ, itaque, ſubſtituto $x$ pro $Q$,

Ponatur $(1+x)^{\frac{m}{n}} = 1 + ax + bx^2 + cx^3 + dx^4$, &c.

Et ſimiliter, aſſumptâ, ad libitum, aliâ quantitate $y$,

Erit $(1+y)^{\frac{m}{n}} = 1 + ay + by^2 + cy^3 + dy^4$, &c.

Et ſubducendo poſteriorem a priore,

$$(1+x)^{\frac{m}{n}} - (1+y)^{\frac{m}{n}} = (ax - ay) + (bx^2 - by^2) + (cx^3 - cy^3) \text{ \&c.}$$

Et dividendo per $(1+x) - (1+y) = x - y$,

$$\frac{(1+x)^{\frac{m}{n}} - (1+y)^{\frac{m}{n}}}{(1+x) - (1+y)} = \frac{a(x-y) + b(x^2-y^2) + c(x^3-y^3), \text{ \&c.}}{x-y}$$

Sive, per *Lemma* [f], ſubſtituendo $1+x$ pro $p$; et $1+y$ pro $q$;

[f] *Lemma* præcedens abſque demonſtratione protulit *Landen*, in Tractatu cui titulus, *A Diſcourſe concerning the Reſidual Analyſis*, *Lond.* 1758. ſpeciminis inſtar, antegreſſo opus ipſum *The Reſidual Analyſis*, anno 1764 impreſſum; ubi inveſtigationem ejus aliter inſtituit auctor, pag. 6. Verùm hujus mihi antehac copia non fuit.



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$$(1+x)^{\frac{m}{n}} - 1 \times \frac{1 + \frac{1+y}{1+x} + \left(\frac{1+y}{1+x}\right)^2 + \left(\frac{1+y}{1+x}\right)^3 \ldots m}{1 + \left(\frac{1+y}{1+x}\right)^{\frac{m}{n}} + \left(\frac{1+y}{1+x}\right)^{\frac{2m}{n}} + \left(\frac{1+y}{1+x}\right)^{\frac{3m}{n}} \ldots\ldots n}$$

$$= a + b(x+y) + c(x^2 + xy + y^2), \text{ \&c.}$$

Sed ſalva æquatione, variari poteſt $y$ ad libitum; fiat igitur $y = x$, et ſingulis terminis tam numeratoris $1 + \frac{1+y}{1+x}$ &c. quam denominatoris $1 + \left(\frac{1+y}{1+x}\right)^{\frac{m}{n}}$ &c. per unitatem jam deſignandis, exponet $m$ ſummam unitatum in numeratore, et $n$ ſummam unitatum in denominatore, et proinde $\frac{m}{n}$ totius ſeriei valorem. Hinc erit,

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}-1} = a + 2bx + 3cx^2 + 4dx^4 \text{ \&c.}$$

Et multiplicando per $1 + x$, erit,

$$\frac{m}{n} \times (1+x)^{\frac{m}{n}} \text{ ſive,}$$

$$\frac{m}{n} \times (1 + ax + bx^2 \text{ \&c. ſive,}$$

$$\frac{m}{n} + \frac{m}{n}ax + \frac{m}{n}bx^2 \text{ \&c.} = a + \frac{2b}{a}x + \frac{3c}{2b}x^2 + \frac{4d}{3c}x^3 \text{ \&c.}$$

Collatis igitur terminis hinc inde homologis, ſcil. $\frac{m}{n} = a$; $\frac{m}{n}ax = (2b+a)x$; &c. eruentur coefficientes,

$$a = \frac{m}{n}.$$

$$b = a \times \frac{m-n}{2n} = \frac{m}{n} \cdot \frac{m-n}{2n}.$$

$$c = b \times \frac{m-2n}{3n} = \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n}.$$

&c. &c.

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Reſtituto jam Q pro $x$; et ſubſtitutis his valoribus $a$, $b$, $c$, &c. in ſeriem $1 + aQ + bQ^2$ &c. emerget,

$$(1+Q)^{\frac{m}{n}} = 1 + \frac{m}{n}Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 \text{ \&c.}$$

Et proinde,

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} \times (1+Q)^{\frac{m}{n}} = P^{\frac{m}{n}} \times (1 + \frac{m}{n}Q + \frac{m}{n} \cdot \frac{m-n}{2n} Q^2 \text{ \&c. five, } = P^{\frac{m}{n}} \times (1 + \frac{m}{n}Q + \frac{\frac{m}{n} \cdot \overline{\frac{m}{n}-1}}{2} Q^2 \text{ \&c.}$$

Hinc quoque, erit,

$$(P-PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} \times (1 - \frac{m}{n}Q + \frac{\frac{m}{n} \cdot \overline{\frac{m}{n}-1}}{2} Q^2 - \frac{\frac{m}{n} \cdot \overline{\frac{m}{n}-1}}{2} \cdot \frac{\overline{\frac{m}{n}-2}}{3} Q^3 + \text{\&c.}$$

mutatis ſcilicet ſignis terminorum alternorum a ſecundo incipientium, quos ingrediuntur impares dimenſiones quantitatis jam negativæ Q.

Cor. I. Creſcente indice Q communi differentiâ 1 in terminis ſucceſſivis, pariter decreſcit index P, ita ut ſumma indicum utriuſque ſingulis in terminis evadat $\frac{m}{n}$; cum Q revera ſit fractio cujus denominator eſt P. Et quo melius forſan conſtet ratio decrementi indicis P, formulâ inſequente theorema utplurimum efferunt recentiores,

$$(p+q)^{\frac{m}{n}} = p^{\frac{m}{n}} + \frac{m}{n}p^{\frac{m}{n}-1}q + \frac{\frac{m}{n} \cdot \overline{\frac{m}{n}-1}}{2} p^{\frac{m}{n}-2} q^2 \text{ \&c.}$$

Ubi deſignat $p$, primum terminum; $q$ ſummam reliquorum.

Nam $(p+q)^{\frac{m}{n}} = p^{\frac{m}{n}} \times \left(1+\frac{q}{p}\right)^{\frac{m}{n}}$

$$= p^{\frac{m}{n}} \times \left(1 + \frac{m}{n}\frac{q}{p} + \frac{\frac{m}{n}\cdot\overline{\frac{m}{n}-1}}{2}\left.\frac{q}{p}\right|^2 \text{ \&c.}\right.$$

$$= p^{\frac{m}{n}} + \frac{m}{n}p^{\overline{\frac{m}{n}-1}}q + \frac{\frac{m}{n}\cdot\overline{\frac{m}{n}-1}}{2}p^{\frac{m}{n}-2}q^2 \text{ \&c.}$$

Cor. II. Si index $\frac{m}{n}$ ſit integer et affirmativus, terminabitur ſeries dimenſionem quæſitam deſignans; ſi non, excurret in infinitum.

Nam ſi $\frac{m}{n}$ ſit integer et poſitivus, ſcil. $\frac{m}{1}=m$; (evadente jam $n=1$.) progreſſio numeratorum $m$, $m-1$, $m-2$, $m-3$, &c. terminis continuatis per $m-m=0$, neceſſario tranſibit, qui idcirco iſtum terminum cujus compoſitionem ingreditur, deſtruet; at hoc termino evaneſcente, evaneſcent quoque ſequentes inde conflati, adeoque rumpetur ſeries. Numerus autem terminorum in hoc caſu erit $m+1$, cum progreſſio $m$, $m-1$, &c. non niſi a ſecundo ſeriei termino auſpicatur, exiſtente unitate primi termini coefficiente. Si $\frac{m}{n}$ ſit integer et negativus, ſcil. $\frac{-m}{1}=-m$; erit $-m-1$ magis negativus, et adhuc magis $-m-2$; adeoque longius a nihilo recedentibus terminis ſequentibus, progreſſio $-m$, $-m-1$, $-m-2$, &c. per nihilum tranſire nequit, adeoque in infinitum excurret ſeries. Hoc quoque ita ſe habebit, ſi $\frac{m}{n}$ ſit fractus; nam ſi affirmativè ponatur, *propè* nihilum ſed non *per* nihilum tranſibit progreſſio $\frac{m}{n}$, $\frac{m}{n}-1$, &c. Ex. gr. ſit $\frac{m}{n}=\frac{1}{2}$; erit $\frac{m}{n}-1=-\frac{1}{2}$; $\frac{m}{n}-2=-\frac{3}{2}$, &c. ſi negativè, longius ut antea, a nihilo recedent termini ſucceſſivi.

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60. Hujus theorematis ope, quantitatis cujuſlibet compoſitæ ſimplices poteſtates eliciuntur. Ut ſi deſideretur quadratum $a + b$ ſive $\overline{a+b}^2$ erit $P = a$; $Q = \frac{b}{a}$; $m = 2$; $n = 1$. adeoque A $\left(= P^{\frac{m}{n}}\right) = a^2$; B $\left(= \frac{m}{n} AQ\right) = 2ab$; C $\left(= \frac{\frac{m}{n} - 1}{2} BQ\right) = b^2$; D $\left(= \frac{\frac{m}{n} - 2}{3} CQ\right) = 0$, cum factor $\frac{m}{n} - 2 = 2 - 2 = 0$. Ergo termino tertio terminabitur ſeries, eritque $\overline{a+b}^2 = a^2 + 2ab + b^2$. Eſt quoque $\overline{a+b}^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. Nam $P = a$; $Q = \frac{b}{a}$; $m = 5$; $n = 1$. Adeoque, A $\left(= P^{\frac{m}{n}}\right) = a^5$; B $\left(= \frac{m}{n} AQ\right) = 5a^4b$; $C = 10a^3b^2$; $D = 10a^2b^3$; $E = 5ab^4$; $F = b^5$ et G $\left(= \frac{\frac{m}{n} - 5}{6} FQ\right) = 0$.

61. Eodem modo extrahuntur radices aut quantitatis ipſius aut quarumlibet ejus dimenſionum. Ut ſi quæratur radix quadratica $a^2 + 2ab + b^2$ ſive $\overline{a^2+2ab+b^2}^{\frac{1}{2}}$, erit $P = a^2$; $Q = \frac{2ab + b^2}{a^2}$; $m = 1$; $n = 2$; adeoque,

$$
\begin{array}{lllll}
A\left(= P^{\frac{m}{n}}\right) & = a & & & \\
B\left(= \frac{m}{n} AQ\right) & = +b + \frac{1b^2}{2a} & & & \\
C\left(= \frac{m-n}{2n} BQ\right) = & -\frac{1b^2}{2a} & -\frac{1b^3}{2a^2} & -\frac{1b^4}{8a^3} & \\
D\left(= \frac{m-2n}{3n} CQ\right) = & & +\frac{1b^3}{2a^2} & +\frac{3b^4}{4a^3} & +\frac{3b^5}{8a^4} + \&c. \\
E\left(= \frac{m-3n}{4n} DQ\right) = & & & -\frac{5b^4}{8a^3} & -\frac{5b^5}{4a^4} - \&c. \\
\&c. & & & & \\
\hline
(a^2 + 2ab + b^2)^{\frac{1}{2}} & = a + b & * & * & * \quad \&c.
\end{array}
$$

Erit igitur $a + b$ radix, quæſita, reliquis terminis ſeſe deſtruentibus, ut æquum eſt, ſi ſeries in infinitum continuatur. Atque hac methodo, radices omnium quantitatum perfectas poteſtates referentium extrahendæ ſunt accurate, aliarum vero quamproxime. Sic radix quadrato cubica $c^5 + c^4 x - x^5$, ſive $(c^5 + c^4 x - x^5)^{\frac{1}{5}} = c + \frac{c^4 x - x^5}{5c^4} - \frac{2c^8 x^2 + 4c^4 x^6 - 2x^{10}}{25c^9}$ + &c. ut patebit ſubſtituendo $c^5$ pro P; $\frac{c^4 x - x^5}{c^5}$ pro Q; 1 pro $m$; et 5 pro $n$. fiet enim $A = c$. $B = \frac{c^4 x - x^5}{5c^4}$ &c. Potuit quoque ſeries ordine inverſo inſtitui; et tunc foret $= -x + \frac{c^4 x + c^5}{5x^4} + \frac{2c^8 x^2 + 4c^9 x + 2c^{10}}{25x^9}$ + &c. Prior modus eligendus eſt, ſi $c$ ſit major quam $x$; poſterior ſi contra, ut convergat ſeries: et quo minor Q quam P, eò citius converget ſeries, ſive paucioribus terminis adhibitis exponetur tota quamproxime. Poſito enim quod Q evaneſcat, primus ſeriei terminus A totam deſignabit, evaneſcentibus ſecundo $\frac{m}{n} AQ$, et reliquis inde genitis; et proinde quò minor ſit Q, eò minor primo termino evadet ſecundus; et ſecundo tertius, &c. Itemque radix cubica ex bi-quadrato $a + b$ ſive $(a + b)^{\frac{4}{3}}$ evadit $a^{\frac{4}{3}} + \frac{4a^{\frac{1}{3}} b}{3} + \frac{2b^2}{9a^{\frac{2}{3}}} - \frac{4b^3}{81a^{\frac{5}{3}}}$ + &c. Nam $P = a$. $Q = \frac{b}{a}$. $m = 4$. $n = 3$. Adeoque, $A \left(= P^{\frac{m}{n}}\right) = a^{\frac{4}{3}}$ &c.

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62. Quinetiam diviſio ſive ſimplex fuerit, ſive per poteſtates aut quantitates radicales, ope hujus theorematis perficitur. Ut ſi $\frac{1}{d+e}$ (hoc eſt $\overline{d+e}\,^{-1}$ ſive $\overline{d+e}\,^{-\frac{1}{1}}$) in ſeriem ſimplicium terminorum reſolvenda ſit; erit $P = d$, $Q = \frac{e}{d}$. $m = -1$. $n = 1$. Adeoque, $A\,(= P^{\frac{m}{n}}) = d^{-1}$ ſeu $\frac{1}{d}$. $B\,(= \frac{m}{n} AQ) = -1 \times \frac{1}{d} \times \frac{e}{d} = -\frac{e}{d^2}$. Et ſic $C = \frac{e^2}{d^3}$. $D = -\frac{e^3}{d^4}$, &c. Hoc eſt, $\frac{1}{d+e} = \frac{1}{d} - \frac{e}{d^2} + \frac{e^2}{d^3} - \frac{e^3}{d^4} +$ &c.

Sic et $\frac{N}{(d+e)^3}$ hoc eſt, N diviſa ter per $d+e$, vel ſemel per cubum ejus, vel $N \times \overline{d+e}\,^{-3} = N \times \frac{1}{d^3} - \frac{3e}{d^4} + \frac{6e^2}{d^5} - \frac{10e^3}{d^6} +$ &c.

Sic $\frac{N}{(y^3-a^2y)^{\frac{1}{3}}}$ hoc eſt, $N \times \overline{y^3-a^2y}\,^{-\frac{1}{3}} = N \times \frac{1}{y} + \frac{a^2}{3y^3} + \frac{2a^4}{9y^5} + \frac{14a^6}{81y^7}$ &c. Nam $P = y^3$. $Q = -\frac{a^2}{y^2}$. $m = -1$. $n = 3$. Adeoque, $A\,(= P^{\frac{m}{n}} = y^{3\times-\frac{1}{3}}) = y^{-1}$ ſeu $\frac{1}{y}$. $B\,(= \frac{m}{n} AQ) = -\frac{1}{3} \times \frac{1}{y} \times -\frac{a^2}{y^2} = \frac{a^2}{3y^3}$ &c.

Itemque, $\frac{\sqrt{c^4 \begin{smallmatrix}-cc\\+rr\end{smallmatrix} x^2}}{\sqrt{c^4 - ccx^2}} = \overline{c^4 \begin{smallmatrix}-cc\\+rr\end{smallmatrix} x^2}\,\Big|^{\frac{1}{2}} \times \overline{c^4 - ccx^2}\,\Big|^{-\frac{1}{2}}$

ſive compendii gratia, ſubſtituto $a$ pro $c^2$ et $b$ pro $-cc+rr$ evadit factor prior,

$$(a^2+bx^2)^{\frac{1}{2}} = a + \frac{bx^2}{2a} - \frac{b^2x^4}{8a^3} + \frac{b^3x^6}{16a^5} - \&c.$$

Et poſterior,

$$(a^2-ax^2)^{-\frac{1}{2}} = \frac{1}{a} + \frac{x^2}{2a^2} + \frac{3x^4}{8a^3} + \frac{5x^6}{16a^4} + \&c.$$

Adeoque,

$$(a^2+bx^2)^{\frac{1}{2}}\times(a^2-ax^2)^{-\frac{1}{2}} = \left\{\begin{array}{l} 1+\frac{bx^2}{2a^2} - \frac{b^2x^4}{8a^4} + \frac{b^3x^6}{16a^6} - \&c. \\ \quad \frac{x^2}{2a} + \frac{bx^4}{4a^3} - \frac{b^2x^6}{16a^5} + \&c. \\ \qquad + \frac{3x^4}{8a^2} + \frac{3bx^6}{16a^4} - \&c. \\ \qquad\quad + \frac{5x^6}{16a^3} + \&c. \end{array}\right.$$

$$= 1 + \frac{a+b}{2a^2}x^2 + \frac{3a^2+2ab-b^2}{8a^4}x^4 + \&c.$$

Adeoque reſtituendo $c^2$ pro $a$ et $r^2-c^2$ pro $b$, evadit,

$$\left(c^4 \begin{array}{l}-cc\\+rr\end{array} x^2\right)^{\frac{1}{2}} \times \left(c^4-ccx^2\right)^{-\frac{1}{2}} = 1 + \frac{r^2}{2c^4}x^2 + \frac{4r^2c^2-r^4}{8c^8}x^4$$

$+ \&c.$

63. Per eandem regulam, in numeris perinde ac in ſpeciebus geneſis poteſtatum, diviſio per poteſtates aut per quantitates radicales, et præſertim extractio radicum altiorum commode inſtituuntur. Nam quiſque numerus ex partibus componi concipiatur, adeoque in binomium æquipollentem reſolvi poteſt.

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64. Ut ſi deſideretur quinta poteſtas numeri 6, reſolvatur 6 in partes 1 et 5, eritque $\overline{1+5}^5 = 7776$. Sit enim $P = 1$, $Q = 5$, $\frac{m}{n} \left(= \frac{5}{1}\right) = 5$ erit,

$$A = P^{\frac{m}{n}} \quad = \quad 1$$
$$B = \frac{m}{n} AQ \quad = \quad 25$$
$$C = \frac{\frac{m}{n} - 1}{2} BQ = 250$$
$$D = \frac{\frac{m}{n} - 2}{3} CQ = 1250$$
$$E = \frac{\frac{m}{n} - 3}{4} DQ = 3125$$
$$F = \frac{\frac{m}{n} - 4}{5} EQ = \underline{3125}$$
$$7776$$

Potuit ſeries ordine inverſo inſtitui, ponendo $P = 5$ et $Q = \frac{1}{5}$; et idem, ut æquum eſt, prodiret valor. Potuit quoque 6 in alias partes reſolvi, 4 et 2; 3 et 3; 11 et $-5$, &c. cæterum facillima prodit involutio ubi P ſit unitas.

65. At ſi quæratur radix quadratica 6, reſolvendus eſt 6 in partes 4 et 2; prior enim debet eſſe numerus quadratus quo rationalis evadet primus ſeriei terminus, alterâ autem

major quo convergat ſeries. Sit itaque $P = 4$, $Q = \frac{1}{2}$, $m = 1$, $n = 2$, eritque, $A\ (= P^{\frac{m}{n}}) = 2$, $B\ (= \frac{m}{n} AQ) = \frac{1}{2}$, $C\ (= \frac{m-n}{2n} BQ) = -\frac{1}{16}$, &c. Cæterum aggregatum duorum terminorum $2 + \frac{1}{2}$ radicem fere deſignabit; nam quadratum hujus ſeu $\frac{5}{2}$ eſt $\frac{25}{4} = 6\frac{1}{4}$, trium vero $2 + \frac{1}{2} - \frac{1}{16} = \frac{39}{16}$, propius; nam hujus quadratum $= \frac{1521}{256} = 6 - \frac{15}{256}$. Et ſic plurium terminorum additione adhuc propius radicem eruere liceret. Præſtat autem de novo ſeriem citius convergentem inſtituere, reſolvendo 6 in partes $\frac{25}{4}$ et $-\frac{1}{4}$. Deſignante enim P $\frac{25}{4}$, Q $-\frac{1}{25}$, erit $A = \frac{5}{2}$, $B = -\frac{1}{20}$, &c. at $\frac{5}{2} - \frac{1}{20} = \frac{49}{20}$ cujus quadratum $\frac{2401}{400} = 6 + \frac{1}{400}$. Et ſi adhuc propius radicem eruere quæramus, inſtitui poteſt ſeries citiſſime convergens, reſolvendo 6 in partes $\frac{2401}{400}$ et $-\frac{1}{400}$ his enim pro P et PQ ſubſtitutis, evadet $A = \frac{49}{20}$, $B = -\frac{1}{1960}$, at $\frac{49}{20} - \frac{1}{1960} = \frac{4801}{1960}$ cujus quadratum prodit $\frac{23049601}{3841600} = 6 + \frac{1}{3841600}$.

Si deſideretur radix cubica 3, reſolvendus eſt 3 in partes duas quarum prior debet eſſe numerus cubicus, ſcil. 8 et $-5$, deſignante enim P, 8; Q, $-\frac{5}{8}$; $m$, 1; $n$, 3, erit

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$A\,(=P^{\frac{m}{n}}) = 2$, $B\,(=\frac{m}{n}AQ) = -\frac{5}{12}$, $C\,(=\frac{m-n}{2n}BQ) = -\frac{25}{288}$, &c. Cæterum $2-\frac{5}{12}-\frac{25}{288}=\frac{431}{288}$ radicem aliquantulum ſuperabit, nam hujus cubus fit $3+\frac{8399375}{23887872}$. Satius igitur foret ſeriem ab initio quæ citius convergeret condere, quærendo radicem cubicam numeri $8 \times 3 = 24$, quæ duplo major prodibit radice cubica ternarii. Reſolvatur itaque 24 in partes 27, et $-3$, et ponendo $P=27$, $Q=\frac{-1}{9}$, erit $A=3$, $B=-\frac{1}{9}$, &c. Cæterum erit $3-\frac{1}{9}=\frac{26}{9}$ radix cubica numeri 24 fere, adeoque hujus ſemiſſis $\frac{13}{9}=\sqrt[3]{3}$ fere, nam hujus cubus fit $\frac{2197}{729}=3+\frac{10}{729}$. Quamproxime autem eruetur radix ope ſeriei citiſſime convergentis, iterum reſolvendo 3 in partes $\frac{2197}{729}$ et $-\frac{10}{729}$, erit enim $A=\frac{13}{9}$, $B=\frac{-10}{4563}$, &c. Sed radicis $\frac{13}{9}-\frac{10}{4563}=\frac{6581}{4563}$ cubus prodit $3+\frac{1976300}{95006081547}$.

Si deſideretur radix quadrato-cubica numeri 33, reſolvendus eſt 33 in partes quarum prior debet eſſe numerus quadrato-cubicus ſcilicet 32 et 1, deſignante enim P, 32; Q, $\frac{1}{32}$; $m$, 1; et $n$, 5. evadet $A=2$, $B=\frac{1}{80}$. radicis autem $2+\frac{1}{80}=\frac{161}{80}$ quinta dimenſio eſt $33+\frac{41216801}{3276800000}$.

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# ANALYSIS ÆQUATIONUM.

## PARS SECUNDA.

### DE ÆQUATIONE FINALI EX DUABUS PLURIBUSVE MEDIIS ELICIENDA.

66. Cum in alicujus problematis ſolutionem plures habentur æquationes ſtatum quæſtionis comprehendentes, quarum unicuique plures etiam quantitates incognitæ involvuntur; æquationes iſtæ, (duæ per vices ſi modo ſint plures duabus) ſunt *ita connectendæ*, ut una ex quantitatibus incognitis per ſingulas operationes tollatur, et tandem emergat æquatio in qua unica manebit quantitas incognita, ſi tot ſunt quantitates incognitæ quot habentur æquationes; ut in problematis determinatis ſolet. Sin quantitates incognitæ ſint unâ plures quam æquationes habentur, tum in æquatione ultimo reſultante duæ manebunt quantitates incognitæ; ſi duabus plures, &c. manebunt tres; et ſic deinceps: ut in problematis indeterminatis. Nam ſciendum eſt, quod per quamlibet æquationem una quantitas incognita tollatur.

67. Quemadmodum *Si quærantur duo numeri* x *et* y *quorum ſumma ſit* a *et differentia* d. Erit $x + y = a$ et $x - y = d$. Adeoque addendo æqualia æqualibus $2x = a + d$ et $x = \frac{1}{2}a + \frac{1}{2}d$. Sive, demendo æqualia ab æqualibus $2y = a - d$ et $y = \frac{a-d}{2}$. Sic quoque, habitis æquationibus $ax = \frac{b^2x - b^3}{z}$ et $x = \frac{az}{x - b}$, multiplicando æqualia æqualibus, prodibit $ax^2 = \frac{ab^2xz - ab^3z}{xz - bz} = ab^2$ ſive $x = b$.



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Cæterum rarò ita contexuntur æquationes ut exterminationem tam facilem et expeditam patiantur quantitates incognitæ quas involvunt, præſertim ſi ſint altiorum dimenſionum. Poſſunt etiam duæ vel plures quantitates incognitæ per duas tantum æquationes *fortaſſe* tolli. Ut ſi ſit $ax - by = ab - az$, et $bx + by = b^2 + az$; tum æqualibus ad æqualia additis prodibit $ax + bx = ab + b^2$, exterminatis utriſque $y$ et $z$: adeoque $x = b$. Sed ejuſmodi caſus vel arguunt vitium aliquod in ſtatu quæſtionis latere, vel calculum erroneum eſſe aut non ſatis artificioſum. Modus autem quo una quantitas incognita per ſingulas æquationes tollatur ex ſequentibus patebit.

## De unica incognita quantitate e duabus æquationibus tollenda.

68. Reg. I. *Cum quantitas tollenda eſt unius tantum dimenſionis in utraque æquatione; valor ejus uterque, per regulas jam ante traditas quærendus eſt, & alter valor ſtatuendus alteri æqualis.*

Hoc idem quoque perficitur (et plerumque facilius) *ſubducendo alterutrum valorem quantitatis incognitæ ab altero, et ponendo reſiduum æquale nihilo.*

69. Reg. II. *Cum quantitas tollenda unius exiſtit dimenſionis in una æquatione, et duarum pluriumve in altera; quærendus eſt valor ejus in priore, & pro ſe in alteram ſubſtituendus.*

70. Reg. III. *Cum quantitas tollenda plurium in utraque æquatione dimenſionum exiſtit; valor maximæ poteſtatis ejus in utraque æquatione quærendus eſt, deinde, ſi poteſtates iſtæ non ſint eædem (et æquatio poteſtatis majoris deprimi nequeat, adeo ut evadat ejuſdem poteſtatis cum æquatione minoris) æquatio poteſtatis*

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*minoris multiplicanda eſt per tollendam quantitatem, aut per ejus quadratum aut cubum, &c. ut ea evadat ejuſdem poteſtatis cum æquatione altera. Tum valores illarum poteſtatum ponendæ ſunt æquales, et æquatio nova prodibit ubi maxima poteſtas ſive dimenſio tollendæ quantitatis diminuitur. Et hanc operationem iterando quantitas illa tandem auferetur.*

71. Harum regularum generalia quædam exempla trademus, quæ tanquam *Formulas* ad minuendum laborem quantitatis multarum dimenſionum e duabus æquationibus tollendæ, abſque ſolutione commendat *Newtonus*. Analogiæ autem gratiâ a caſu ſimpliciſſimo ordiri libet, quo melius conſtet creſcente quantitatis incognitæ dimenſione, quantum creſcit exterminandi labor. Porro, notandum eſt, literis initialibus $a$, $b$, $c$, $d$, &c. in his formulis deſignari tum quantitates cognitas, tum quaſcunque poteſtates alterius cujuſdam quantitatis incognitæ.

FORMULA I.

$$ax + b = 0.$$
$$fx + g = 0.$$

Per REG. I. Erit, $\frac{b}{a} (= -x) = \frac{g}{f}$ adeoque, $bf - ag = 0$.

FORMULA II.

$$ax^2 + bx + c = 0.$$
$$fx + g = 0.$$

Per REG. II. Subſtituto $\frac{-g}{f}$ pro $x$ in æquationem priorem,

Erit, $\frac{ag^2}{f^2} - \frac{bg}{f} + c = 0$.

Adeoq. $ag^2 - bfg + cf^2 = 0$.

Sive, $(bf - ag)(g) - cf^2 = 0$.

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## Formula III.

$$ax^2 + bx + c = 0.$$
$$fx^2 + gx + h = 0.$$

Per Reg. III. Erit, $\frac{bx + c}{a} (= - x^2) = \frac{gx + h}{f}$

Adeoque, $x = \frac{ah - cf}{bf - ag}.$

Et, $- x^2 = \frac{(- ah + cf)x}{bf - ag}.$

Adeoque, $\frac{bx + c}{a} (= - x^2) = \frac{(- ah + cf)x}{bf - ag}.$

Et, $x = \frac{- bcf + acg}{b^2 f - abg + a^2 h - acf}$

Adeoque, $\frac{ah - cf}{bf - ag} (= x) = \frac{- bcf + acg}{b^2 f - abg + a^2 h - acf}.$

Unde concinnando, et ordinando erit

$$a^2 h^2 - abgh - 2acfh : + b^2 fh - bcfg : + acg^2 + c^2 f^2 = 0.$$

Et reſolvendo in factores *Newtonianos*,

$$(ah - bg - 2cf)(ah) : + (bh - cg)(bf) : + (ag^2 + cf^2)(c) = 0.$$

Idem quoque perfici potuit ſubſtituendo per Reg. II. unum ex valoribus $x$, ſcilicet $\frac{ah - cf}{bf - ag}$ in alterutram æquationum propoſitarum, puta priorem, nam ex $a \times \left(\frac{ah - cf}{bf - ag}\right)^2 + b \times \frac{ah - cf}{bf - ag} + c = 0.$ concinnando et ordinando eadem prodibit concluſio.

72. Simili ratione, at calculo maxime laborioſo, quantitatem tollendam altiorum dimenſionum eliminare poſſimus. Verùm longe concinnius expedietur exterminatio ope regulæ inſequentis.

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73. REG. IV. *Multiplicetur viciſſim utraque æquatio eundem referens aut ad eundem reducta gradum, per coefficientem termini altiſſimi in altera, tum ſubducendo alterutram æquationem ab altera, tolletur poteſtas maxima quantitatis incognitæ, et emerget nova et depreſſior æquatio. Multiplicetur iterum viciſſim utraque æquatio per alterius terminum abſolutum ſeu ultimum, tum ſubducendo, &c. tolletur utriuſque terminus ultimus, et dividendo inſuper per incognitam quantitatem alia emerget æquatio ad eandem dimenſionem depreſſa. Deinde, collatis his novis æquationibus, eandem operationem iterando tandem tolletur* x.

Ad normam hujus jam expediamus ſolutionem Formulæ tertiæ, eritque

$$afx^2 + bfx + cf = 0 \qquad \text{et } abx^2 + bbx + cb = 0.$$
$$afx^2 + agx + ab = 0 \qquad cfx^2 + cgx + cb = 0.$$
$$* \ (bf - ag)x + cf - ab = 0 \qquad (cf - ab)x + (cg - bb^* = 0.$$

$$(bf - ag)(cf - ab)x + (cf - ab)^2 = 0.$$
$$(bf - ag)(cf - ab)x + (bf - ag)(cg - bb) = 0.$$
$$* \ (cf - ab)^2 - (bf - ag)(cg - bb) = 0.$$

Sive, $c^2f^2 - 2acfb + a^2b^2 - bcfg + b^2fb + acg^2 - abgb = 0.$

## FORMULA IV.

$$ax^3 + bx^2 + cx + d = 0.$$
$$fx^2 + gx + b = 0.$$

$$afx^3 + bfx^2 + cfx + df = 0.$$
$$afx^3 + agx^2 + abx = 0.$$
$$* \ (bf - ag)x^2 + (cf - ab)x + df = 0.$$ Sive abbreviandi gratiâ,
$$Ax^2 + Bx + df = 0.$$



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$$Afx^2 + Bfx + df^2 = 0. \qquad Abx^2 + Bbx + dfb = 0.$$
$$Afx^2 + Agx + Ab = 0. \qquad df^2x^2 + dfgx + dfb = 0.$$
$$*(Bf - Ag)x + df^2 - Ab = 0. \qquad (df^2 - Ab)x + (dfg - Bb * = 0.$$

$$(Bf - Ag)(df^2 - Ab)x + (df^2 - Ab)^2 = 0.$$
$$(Bf - Ag)(df^2 - Ab)x + (Bf - Ag)(dfg - Bb) = 0.$$
$$* (df^2 - Ab)^2 - (Bf - Ag)(dfg - Bb) = 0. \text{ ſive,}$$

$$d^2f^4 - 2Adf^2h + A^2h^2 - Bdf^2g + B^2fh + Adfg^2 - ABgh = 0;$$
$$= d^2f^3 - 2Adfh - Bdfg + B^2h + Adg^2 + \frac{A^2h^2 - ABgh}{f} = 0.$$

Reſtitutis jam $bf - ag$ pro A, et $cf - ah$ pro B, prodit,

$$\left.\begin{array}{l} d^2f^3 - 2bdf^2h + 3adfgh - cdf^2g + bdfg^2 - adg^3 + c^2f^2h \\ - 2acfh^2 + a^2h^3 + b^2fh^2 - abgh^2 - bcfgh + acg^2h \ \ldots \end{array}\right\} = 0.$$

Sive ſecundum formulam *Newtonianam*,

$$\left.\begin{array}{l} (ah - bg - 2cf)\, ah^2 : + (bh - cg - 2df)\, bfh : \\ + (ch - dg)(ag^2 + cf^2) : + (3agh + bg^2 + df^2)\, df \end{array}\right\} = 0.$$

## Formula V.

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$
$$fx^2 + gx + h = 0.$$

$$afx^4 + bfx^3 + cfx^2 + dfx + ef = 0.$$
$$afx^4 + agx^3 + ahx^2 = 0.$$
$$* (bf - ag)x^3 + (cf - ah)x^2 + dfx + ef = 0.$$

Sive, $\qquad Ax^3 + Bx^2 + dfx + ef = 0.$

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$$Afx^3 + Bfx^2 + df^2x + ef^2 = 0.$$
$$Afx^3 + Agx^2 + Ahx = 0.$$

$$* \, (Bf - Ag)\,x^2 + (df^2 - Ah)\,x + ef^2 = 0.$$

Sive, $A'x^2 + B'x + ef^2 = 0.$

$$A'fx^2 + B'fx + ef^3 = 0. \qquad A'hx^2 + B'hx + ef^2h = 0.$$
$$A'fx^2 + A'gx + A'h = 0. \qquad ef^3x^2 + ef^2gx + ef^2h = 0.$$

$$* (B'f - A'g)x + ef^3 - A'h = 0. \qquad (ef^3 - A'h)\,x + ef^2g - B'h \, * = 0.$$

$$(B'f - A'g)\,(ef^3 - A'h)\,x + (ef^3 - A'h)^2 = 0.$$
$$(B'f - A'g)\,(ef^3 - A'h)\,x + (B'f - A'g)\,(ef^2g - B'h) = 0.$$

$$* \; (ef^3 - A'h)^2 - (B'f - A'g)\,(ef^2g - B'h) = 0.$$

$$e^2f^6 - 2A'ef^3h + A'^2h^2 - B'ef^3g + A'ef^2g^2 + B'^2fh - A'B'gh = 0.$$

five $e^2f^5 - 2A'ef^2h - B'ef^2g + A'efg^2 + B'^2h + \frac{A'^2h^2 - A'B'gh}{f} = 0.$

Reſtitutis valoribus $A'$ et $B'$ ſcilicet $Bf - Ag$ et $df^2 - Ah$ erit,

$$\left.\begin{aligned} & e^2f^5 - 2Bef^3h + 3Aef^2gh - def^4g + Bef^2g^2 - Aefg^3 \\ & + d^2f^4h - 2Adf^2h^2 + A^2h^3 + B^2fh^2 - ABgh^2 \\ & - Bdf^2gh + Adfg^2h \end{aligned}\right\} = 0.$$

Reſtitutis iterum valoribus A et B, ordinando et dividendo per $f$, erit,

$$\left.\begin{aligned} & a^2h^4 - abgh^3 - 2acfh^3 + b^2fh^3 - bcfgh^2 - 2bdf^2h^2 \\ & + acg^2h^2 - adg^3h + aeg^4 - 4aefg^2h + c^2f^2h^2 - cdf^2gh \\ & + cef^2g^2 - 2cef^3h + 3adfgh^2 + bdfg^2h + d^2f^3h \\ & + 2aef^2h^2 + 3bef^2gh - def^3g + e^2f^4 - befg^3 \end{aligned}\right\} = 0.$$

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Unde facile elicitur formula *Newtoniana*,

$$\left.\begin{array}{l}(ah - bg - 2cf)\,ah^3 : + (bh - cg - 2df)\,bfh^2 : + (ag^2 \\ + cf^2)\,(ch^2 - dgh + eg^2 - 2efh) : + (3agh + bg^2 \\ + df^2)\,dfh : + (2ah^2 + 3bgh - dfg + ef^2)ef^2 : - (bg \\ - 2ah)\,efg^2 \quad - \quad - \quad - \quad - \quad - \quad - \quad - \end{array}\right\} = 0.$$

Cæterum obſervandum eſt terminum $- 4aefg^2h$ ſupra memoratum a *Newtono* in binos reſolvi, ſcilicet $ag^2 \times - 2efh$ & $- 2ah \times efg^2$.

## Formula VI.

$$ax^3 + bx^2 + cx + d = 0.$$
$$fx^3 + gx^2 + hx + k = 0.$$

$$afx^3 + bfx^2 + cfx + df = 0. \qquad akx^3 + bkx^2 + ckx + dk = 0.$$
$$afx^3 + agx^2 + ahx + ak = 0. \qquad dfx^3 + dgx^2 + dhx + dk = 0.$$

$$*(bf - ag)x^2 + (cf - ah)x + df - ak = 0. \qquad (df - ak)x^2 + (dg - bk)x + dh - ck* = 0.$$

Sive, $Ax^2 + Bx + C = 0.$ $\qquad Cx^2 + Dx + F = 0.$

$$ACx^2 + BCx + C^2 = 0. \qquad AFx^2 + BFx + CF = 0.$$
$$ACx^2 + ADx + AF = 0. \qquad C^2x^2 + CDx + CF = 0.$$

$$*(BC - AD)x + C^2 - AF = 0. \qquad (C^2 - AF)x + CD - BF* = 0.$$

$$(BC - AD)(C^2 - AF)x + (C^2 - AF)^2 = 0.$$
$$(BC - AD)(C^2 - AF)x + (BC - AD)(CD - BF) = 0.$$

$$*(C^2 - AF)^2 - (BC - AD)(CD - BF) = 0.$$

Sive, $C^3 - 2ACF - BCD + B^2F + AD^2 + \dfrac{A^2F^2 - ABDF}{C} = 0.$

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Sive reſtitutis valoribus A, B, C, D, F, erit,

$$\left.\begin{array}{l} d^3f^3 - 3ad^2f^2k + 3a^2dfk^2 - a^3k^3 - 2bd^2f^2h + 3ad^2fgh \\ + abdfhk - 3a^2dghk + 3bcdf^2k - acdfgk - 3abcfk^2 \\ + 2a^2cgk^2 - cd^2f^2g + a^2bhk^2 + c^2df^2h - 2acdfh^2 + a^2dh^3 \\ - c^3f^2k + 2ac^2fhk - a^2ch^2k + bd^2fg^2 - 2b^2dfgk + b^3fk^2 \\ - ad^2g^3 + 2abdg^2k - ab^2gk^2 + b^2dfh^2 - abdgh^2 - b^2cfhk \\ + abcghk - bcdfgh + acdg^2h + bc^2fgk - ac^2g^2k \end{array}\right\} = 0.$$

Unde ſi ordine debito diſponantur termini elicietur formula *Newtoniana*,

$$\left.\begin{array}{l} (ah - bg - 2cf)\,(adh^2 - achk) : + (ak + bh - cg - 2df) \\ bdfh : (- ak + bh + 2cg + 3df)\,a^2k^2 : + (cdh - d^2g \\ - c^2k + 2bdk)\,(ag^2 + cf^2) : + (3agh + bg^2 + df^2 - 3afk) \\ d^2f : (- 3ak - bh + cg + df)\,bcfk : + (bk - 2dg)\,b^2fk : \\ (- b^2k - 3adh - cdf)\,agk \quad - \quad - \quad - \quad - \end{array}\right\} = 0.$$

Cæterum notandum eſt terminum $3bcdf^2k$ a *Newtono* in binos reſolvi, ſcilicet $2bdk \times cf^2$ & $df \times bcfk$.

## Formula VII.

$$ax^4 + bx^3 + cx^2 + dx + e = 0.$$
$$fx^4 + gx^3 + hx^2 + kx + l = 0.$$

$$afx^4 + bfx^3 + cfx^2 + dfx + ef = 0. \qquad alx^4 + blx^3 + clx^2 + dlx + el = 0$$
$$afx^4 + agx^3 + ahx^2 + akx + al = 0. \qquad efx^4 + egx^3 + ehx^2 + ekx + el = 0$$

$$(bf - ag)x^3 + (cf - ah)x^2 + (df - ak)x + ef - al = 0. \qquad (ef - al)x^3 + (eg - bl)x^2 + (eh - cl)x + ek - dl = 0.$$

ſive, $Ax^3 + Bx^2 + Cx + D = 0. \qquad Dx^3 + Fx^2 + Gx + H = 0.$

$$ADx^3 + BDx^2 + CDx + D^2 = 0. \qquad AHx^3 + BHx^2 + CHx + DH = 0.$$
$$ADx^3 + AFx^2 + AGx + AH = 0. \qquad D^2x^3 + DFx^2 + DGx + DH = 0.$$
$$*(BD - AF)x^2 + (CD - AG)x + D^2 - AH = 0. \qquad (D^2 - AH)x^2 + (DF - BH)x + DG - CH* = 0$$

ſive, $A'x^2 + B'x + C' = 0. \qquad C'x^2 + D'x + F' = 0.$

$$A'C'x^2 + B'C'x + C'^2 = 0. \qquad A'F'x^2 + B'F'x + C'F' = 0.$$
$$A'C'x^2 + A'D'x + A'F' = 0. \qquad C'^2x^2 + C'D'x + C'F' = 0.$$
$$*(B'C' - A'D')x + C'^2 - A'F' = 0. \qquad (C'^2 - A'F')x + C'D' - B'F'* = 0.$$

$$(B'C' - A'D')(C'^2 - A'F')x + (C'^2 - A'F')^2 = 0.$$
$$(B'C' - A'D')(C'^2 - A'F')x + (B'C' - A'D')(C'D' - B'F') = 0.$$
$$*(C'^2 - A'F')^2 - (B'C' - A'D')(C'D' - B'F') = 0.$$

Sive,

$$C'^4 - 2A'C'^2F' + A'^2F'^2 - B'C'^2D' + B'^2C'F' + A'C'D'^2 - A'B'D'F' = 0.$$

Sive,

$$C'^2 - 2A'F' - B'D + \frac{B'^2C'F' + A'C'D'^2 + A'^2F'^2 - A'B'D'F'}{C'^2} = 0.$$

75. Hujus reductione, ſubſtitutis nimirum valoribus $A'$, $B'$, $C'$, $D'$ et $F'$ tandem eruere licet æquationem 8 dimenſionum quæ formulam quæſitam præ ſe feret. Simili ratione inſtituenda eſt eliminatio quantitatis $x$ ex æquationibus altiorum dimenſionum. In genere autem notandum eſt, quod dimenſio æquationis ultimo reſultantis tranſcendere non debet ſummam dimenſionum æquationum propoſitarum. Huc etenim redit hæc eliminatio, ut inveniatur æquatio nova cujus radix quælibet ſeu valor quantitatis $x$ in alterutram æquationem datam ſubſtituta eam evaneſcere faciet; at ſi dimenſio æquationis finalis hac methodo comparandæ hanc ſummam excedat, ex radicibus ejus nonnullæ finem quæſitum non aſſequentur, radicibus in-

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utilibus unà cum utilibus, admiſtis. Inutiles igitur radices, deprimendo æquationem ad dimenſionem debitam ope diviſoris alicujus ſemper in æquatione finali latentis, reciduntur. Sic in formula VI aſcendit ad octavum gradum æquatio finalis $C^4 - 2AC^2F$, &c. $= 0$, dividendo autem per quadraticam C ſive $df - ak$, $= 0$, ad ſextum deprimitur. In noviſſima item formula, æquatio $C'^4 - 2A'C'^2F'$, &c. $= 0$. 16 dimenſionum, ad æquationem 8 dimenſionum deprimenda eſt ope diviſoris $C'^2$ ſive $\overline{(ef - al)^2 - (bf - ag)(ek - dl)}^2 = 0$. Et quo altius aſſurgunt dimenſiones æquationum propoſitarum eo grandior adhibendus eſt diviſor, et proinde eo moleſtior exterminandi labor. Hujus igitur minuendi gratiâ varias excogitarunt exterminandi methodos Viri Clari, *Cramer*, *Introduction a l'Analyſe des lignes Courbes Algebriques*; *Euler*, *Acad. Berlin*, *anno* 1764; *Bezout*, *Memoirs de l'Acad. Paris*, *anno* 1764; et *La Grange*, *Acad. Berlin* 1769.

Formulis his *Newtonianis* jam inveſtigatis, (exemplis quidem in exercitationem tyronum probe accommodatis) quorundam problematum ſolutiones adjungere libet, quo melius conſtet regularum necnon formularum uſus.

## P R O B. I.

76. *Invenire duos numeros* x *et* y, *quorum ſumma ſit* s *et productum* p.

Habitis duabus æquationibus $x + y = s$, et $xy = p$, ſubſtituatur valor $x$ ex priore reſultans $s - y$ in poſteriorem, et prodibit $\overline{s - y} \times y = p$. Adeoque, $y^2 = sy - p$. Et per reductionem, $y = \frac{1}{2} s \pm \sqrt{\frac{1}{4} s^2 - p}$. Ut ſi $s$ ſit 140; et $p$, 3136; ſubſtituendo prodibit $y = 70 \pm 42 = 28$ vel 112, quorum alterutrum deſignante $y$ alterum deſignabit $x$.

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Exterminari potuit $x$ ex æquationibus $x+y-s=0$ et $xy-p=0$, ope Formulæ I. ſubſtituendo pro $a$, $b$; $f$, $g$; $1$, $y-s$; $y$ et $-p$ reſpective: prodit enim $(\overline{y-s}\times y)-(1\times -p)=0$. ſeu $y^2=sy-p$ ut ſupra.

77. Hinc liquet, quod propoſitis duabus quantitatibus incognitis methodus deſignandi alteram ex aliqua quæſtionis conditione, in ſolvendis problematis antecedentibus adhibita, eodem redit ac exterminatio ſpeciei hanc quantitatem incognitam referentis, eliciendo valorem ejus ex una æquatione eundemque in alteram ſubſtituendo.

## PROB. II.

78. *Invenire ſummam quadratorum, cuborum, biquadratorum, &c. duorum numerorum quorum ſumma ſit* s *et productum* p.

Deſignantibus $x$ et $y$ numeros quæſitos, erit,

$x+y=s$. Adeoque quadrando, $x^2+2xy+y^2=s^2$

et $xy=p$. Subduc - - $2xy=2p$.

Erit *ſumma quadratorum* $x^2+y^2=s^2-2p$.

Multiplicando hanc æquationem per $x+y=s$,

Prodit - $x^3+(x+y)\,xy+y^3=s^3-2sp$

Subduc - $(x+y)\,xy=sp$

Eritque *ſumma cuborum* $x^3+y^3=s^3-3sp$.

Multiplicando iterum per $x+y=s$,

Prodit $x^4+(x^2+y^2)\,xy+y^4=s^4-3s^2p$

Subduc $(x^2+y^2)\,xy=(s^2-2p)p$

Eritque *ſumma biquad.* $x^4+y^4=s^4-4s^2p+2p^2$.

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Hinc innoteſcit lex progreſſionis, ſi quæratur ſumma poteſtatum altiorum; ſemper enim invenietur multiplicando ſummam poteſtatum ultimo inventarum per $s$, et ſubducendo a producto ſummam penultimarum in $p$ ductarum. Et ſic univerſaliter: erit ſumma poteſtatum $n =$

$$s^{n} - ns^{n-2}p + n.\frac{n-3}{2}.s^{n-4}p^{2} - .\frac{n-4}{2}.\frac{n-5}{3}.s^{n-6}p^{3} + n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}.s^{n-8}p^{4}, \&c.$$

## P R O B. III.

79. *Invenire ſummam quadratorum, cuborum, biquadratorum, &c. ex ſingulis quantitatibus* a, b, c, d, *&c. ordinis cujuſcunque, quarum ſumma* a + b + c + d, *&c. ſit* P; *ſumma rectangulorum ſub ſingulis binis*, ab + ac + ad, *&c.* bc + bd, *&c.* + cd, *&c. ſit* Q; *ſumma contentorum ſub ſingulis ternis*, abc + abd + abe, *&c.* + acd + ace, *&c.* + bcd, *&c. ſit* R, *&c. &c.*

Deſignantibus

$p$, ſummam quantitatum $b$, $c$, $d$, &c. poſt primam $a$.
$q$, ſummam rectangulorum $bc + bd$, &c. $+ cd$, &c. poſt $pa$.
$r$, ſummam contentorum $bcd$, &c. $+ cde$, &c. poſt $qa$.
&c. &c.

Erit

$P = a + p$. Adeoque, quadrando, - - $P^2 = a^2 + 2pa + p^2$
$Q = pa + q$. Subduc - - - $2Q = 2pa + 2q$.
$R = qa + r$. Eritque *ſumma quadrat.* - $P^2 - 2Q = a^2 + p^2 - 2q$
$S = ra + s$. Multiplicando hanc æquationem per $P = a + p$.
&c. &c. Erit $P^3 - 2PQ = a^3 + p^3 + pa^2 + p^2a - 2qa - 2pq$
Subduc $PQ = pa^2 + p^2a + qa + pq$

$P^3 - 3PQ = a^3 + p^3 - 3qa - 3pq$
Adde $3R = + 3qa + 3r$

Eritque *Summa cuborum* $P^3 - 3PQ + 3R = a^3 + p^3 - 3pq + 3r$.

Multiplicando hanc per $P = a + p$,

Q

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Erit $P^4 - 3P^2Q + 3PR = a^4 + p^4 + pa^3 + p^3a - 3pqa + 3ra - 3p^2q + 3pr$

Subduc $P^2Q - 2Q^2 = pa^3 + p^3a - 2pqa + qa^2 + p^2q - 2q^2$

$P^4 - 4P^2Q + 3PR + 2Q^2 = a^4 + p^4 - qa^2 - pqa + 3ra - 4p^2q + 3pr + 2q^2$

Adde $PR = qa^2 + pqa + ra + pr$

$P^4 - 4P^2Q + 4PR + 2Q^2 = a^4 + p^4 + 4ra - 4p^2q + 4pr + 2q^2$

Subduc $4S = 4ra + 4s$

Eritque *ſumma biquadratorum*

$$P^4 - 4P^2Q + 4PR + 2Q^2 - 4S = a^4 + p^4 - 4p^2q + 4pr + 2q^2 - 4s.$$

Similiter, præcedentes operationes continuando donec omnes exterminentur dimenſiones $a$ præter quæſitam, prodibit *ſumma quintarum poteſtatum*, $P^5 - 5P^3Q + 5P^2R + 5PQ^2 - 5PS - 5QR + 5T$. Et ſic deinceps. Cæterum combinando valores inter operandum inventos, contractiùs exponere licet ſummas poteſtatum quæſitarum; Ut eleganter *Newtonus*, qui haſce formulis inſequentibus primus abſque demonſtratione propoſuit. Scilicet

$P$ - - - - - - - - - $= A$

$P^2 - 2Q$ - - - - - - $= PA - 2Q = B$

$P^3 - 3PQ + 3R$ - - - $= PB - QA + 3R = C.$

$P^4 - 4P^2Q + 4PR + 2Q^2 - 4S = PC - QB + RA - 4S = D.$

$P^5 - 5P^3Q +$ &c. - - $= PD - QC + RB - SA + 5T = E.$

$P^6 - 6P^4Q +$ &c. - - $= PE - QD + RC - SB + TA - 6V = F.$

Et ſic in infinitum, obſervatâ lege progreſſionis. Neceſſe autem eſt ut probe obſerventur ſigna quantitatum $a$, $b$, $c$, &c. ſi modo diverſis afficiantur ſignis. Has omnes affirmativè poſuimus, hoc tantum innuentes, eas, ſecundum ſigna ſingulis propria eſſe addendas ſeu aggregandas, ſive hæc ſint affirmativa ſive negativa.

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Concluſiones præcedentes verè exponere ſummam quadratorum, cuborum, &c. ita per inductionem conſtabit.

$$P^2 - 2Q = a^2 + b^2 + c^2 + d^2, \text{ \&c.}$$

*Caſ.* 1. Datis tribus quantitatibus $a, b, c$, erit $p = b + c$. Sed per *Prob.* II. $p^2 - 2q = b^2 + c^2$. Ergo, $P^2 - 2Q$ $(= a^2 + p^2 - 2q) = a^2 + b^2 + c^2$.

*Caſ.* 2. Datis quatuor, erit $p = b + c + d$. Sed per *Caſ.* 1. $p^2 - 2q = b^2 + c^2 + d^2$. Ergo, $P^2 - 2Q\ (= a^2 + p^2 - 2q) = a^2 + b^2 + c^2 + d^2$.

*Caſ.* 3. Datis quinque, erit $p = b + c + d + e$. Sed per *Caſ.* 2. $p^2 - 2q = b^2 + c^2 + d^2 + e^2$. Ergo, &c. Et ſic deinceps.

$$P^3 - 3PQ + 3R = a^3 + b^3 + c^3 + d^3, \text{ \&c.}$$

*Caſ.* 1. Datis tribus quantitatibus, erit $p = b + c$. Sed per *Prob.* II. $p^3 - 3pq = b^3 + c^3$. Ergo, $P^3 - 3PQ$ $(= a^3 + p^3 - 3pq) = a^3 + b^3 + c^3$. Ab hoc caſu nimirum ablegatis 3R et $3r$.

*Caſ.* 2. Datis quatuor, erit $p = b + c + d$. Sed,

$$\begin{array}{rl} p^3 = & b^3 + c^3 + d^3 + 3b^2c + 3bc^2 + 3b^2d + 3c^2d + 3bd^2 + 3cd^2 + 6bcd. \\ -3pq = & \qquad\qquad - 3b^2c - 3bc^2 - 3b^2d - 3c^2d - 3bd^2 - 3cd^2 - 9bcd. \\ +3r = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 3bcd. \\ \hline \end{array}$$

Sive, $p^3 - 3pq + 3r = b^3 + c^3 + d^3$.
Ergo, $P^3 - 3PQ + 3R = a^3 + b^3 + c^3 + d^3$.

*Caſ.* 3. Datis quinque, erit $p = b + c + d + e$. Sed per *Caſ.* 2. $p^3 - 3pq + 3r = b^3 + c^3 + d^3 + e^3$. Ergo, &c. Et ſic deinceps. Simili quoque argumentatione in poteſtatibus altioribus conſtabit.

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80. Brevius quidem et elegantius, cæterum argumento paulo ſubtiliore idem colligit *Simpſon*, *Algebra. p.* 129. Quoniam enim ſingulæ quantitates $a$, $b$, $c$, $d$, &c. pariter verſantur in æquationibus originalibus ſeu datis $P = a + b + c$, &c. $Q = abc$, &c. neceſſe eſt, ut pariter quoque verſentur in concluſionibus inde elicitis; adeoque, ſi valores $p$ et $q$ actu ſubſtituantur in æquationem $P^2 - 2Q = a^2 + p^2 - 2q$, liquet nullas reſtare poſſe dimenſiones quantitatum $a$, $b$, $c$, &c. præter quadrata, ex eo quod nulla dimenſio quantitatis $a$ præter quadratum in hâc æquatione occurrit. Et ſimili argumento colligi poteſt nullas dimenſiones quantitatum $a$, $b$, $c$, &c. præter cubos in æquatione $P^3 - 3PQ + 3R = a^3 + p^3 - 3pq + 3r$ verſari. Atque ita deinceps.

## P R O B. IV.

81. *Invenire tres numeros continue proportionales, quorum ſumma ſit* $20 = a$, *et quadratorum ſumma* $140 = b$.

Habitis æquationibus $x + y + \frac{y^2}{x} = 20$, et $x^2 + y^2 + \frac{y^4}{x^2} = 140$, ſubſtituatur in poſteriorem pro $\frac{y^4}{x^2}$ valor ejus ex priore comparandus, ſcilicet $(20 - x - y)^2$, et concinnando prodibit $x^2 + y^2 + \overline{y - 20}\,x - 20y = -130$. Subducatur hæc æquatio ab $x^2 + y^2 + \overline{y - 20}\,x = 0$, ad quam utique revocanda eſt prior, et manebit $20y = 130$ ſeu $y = 6\frac{1}{2}$.

Invento jam $y$, ut inveniatur $x$, ſubſtituatur $6\frac{1}{2}$ pro $y$ in æquationem $x + y + \frac{y^2}{x} = 20$. Et concinnando prodibit

$x^2 = 13\frac{1}{2}\,x - 42\frac{1}{4}$. Et per reductionem $x = 6\frac{3}{4} \pm \sqrt{45\frac{9}{16} - 42\frac{1}{4}} = 6\frac{3}{4} \pm \sqrt{3\frac{5}{16}}$. Nempe $6\frac{3}{4} + \sqrt{3\frac{5}{16}}$ eſt maxi-

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mus quæſitorum trium numerorum, et $6\frac{3}{4} - \sqrt{3\frac{5}{16}}$ minimus. Nam $x$ alterutrum extremorum numerorum ambique deſignat, indeque gemini prodeunt valores, quorum alteruter poteſt eſſe $x$, exiſtente altero $\frac{y^2}{x}$; cum in ſerie proportionalium vel creſcente vel decreſcente conſtituetur $x$, indeque in priore caſu deſignabit maximum proportionalium, in poſteriore minimum.

82. Exterminari potuit $x$ ex æquationibus $x^2 + \overline{y - 20}\, x + y^2 = 0$, et $x^4 + \overline{y^2 - 140}\, x^2 + y^4 = 0$, ad quas redigendæ ſunt præcedentes, ſubſtituendo in *Formulam* V, pro $a$, $b$, $c$, $d$, $e$; $f$, $g$, & $b$: $1, 0, y^2 - 20, 0, y^4$; $1, y - 20$, & $y^2$, reſpective. Et, propter defectum termini ſecundi et quarti in æquatione poſteriore, evaneſcentibus $b$ & $d$ necnon formulæ terminis quorum factores ſunt $b$ & $d$, emerget $(-y^2 + 280)\, y^6 : * + (2y^2 - 40y + 260)\,(260\, y^4 - 40\, y^5) : * + (3y^4)\, y^4 : -2\, y^2\,(y^6 - 40\, y^5 + 400\, y^4) = 0$. Unde multiplicando et delendo ſuperflua, prodibit $1600\, y^6 - 20800\, y^5 - 67600\, y^4 = 0$, ac reducendo, $4y^2 - 52y + 169 = 0$. Sive radice extractâ, $2y - 13 = 0$, ſeu $y = 6\frac{1}{2}$. Ut ſupra. Veruntamen exterminatio quantitatis incognitæ plurium dimenſionum plerunque operoſior evadit formularum ope, quam ſi ab ipſis regulis auſpicemur.

*Sunt et alii modi quibus hæc eadem abſolvi poſſunt; idque ſæpenumero contractius.*

83. Ut ſi generaliter enuncietur quæſtio, habitis æquationibus $x + y + \frac{y^2}{x} = a$, et $x^2 + y^2 + \frac{y^4}{x^2} = b$, prior tranſponendo dat $a - y = x + \frac{y^2}{x}$. Et partibus quadratis $a^2 - 2ay$

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$+ y^2 = x^2 + 2y^2 + \frac{y^4}{x^2}$. Et ſubducendo utrinque $y^2$, erit $a^2 - 2ay \left(= x^2 + y^2 + \frac{y^4}{x^2}\right) = b$. Adeoque $y = \frac{a - \frac{b}{a}}{2}$. Quo ſubſtituto pro $y$ in æquationem $x^2 = \overline{a - y}\, x - y^2$, erit $x^2 = \frac{a + \frac{b}{a}}{2} x - \frac{a^2 - 2b + \frac{b^2}{a^2}}{4}$. Et per reductionem,

$$x = \frac{a + \frac{b}{a} \pm \sqrt{-3a^2 + 10b - \frac{3b^2}{a^2}}}{4}.$$

*Aliter.*

84. Deſignantibus jam $x$ et $y$ proportionales extremos, medium deſignabit $\sqrt{xy}$. Eritque, $x + \sqrt{xy} + y = a$, & $x^2 + xy + y^2 = b$. Dividatur ſecunda æquatio per primam et prodibit $x - \sqrt{xy} + y = \frac{b}{a}$. Subducatur hæc a prima, eritque $2\sqrt{xy} = a - \frac{b}{a}$; et medius $\sqrt{xy} = \frac{a}{2} - \frac{b}{2a}$. Ut ſupra. Unde cognoſcatur ſumma et productum extremorum $x$ et $y$, adeoque & ipſi extremi per *Prob.* I.

## PROB. V.

85. *Invenire quatuor numeros continue proportionales, quorum datur ſumma* a, *& ſumma quadratorum* b.

Deſignantibus $x$ et $y$ medios numeros quæſitos, deſignabunt, ex natura continuè proportionalium, $\frac{x^2}{y}$ et $\frac{y^2}{x}$ extremos. Habentur itaque æquationes $\frac{x^2}{y} + x + y + \frac{y^2}{x} = a$,

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et $\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = b$. Ponamus jam $x + y = s$ et $xy = p$; eritque $\frac{x^2}{y} + \frac{y^2}{x} = a - s$, itemque $\frac{x^2y^2}{xy} = p$ propter proportionalitatem. Sed per *Prob.* II.

Erit ſumma quadrat. $x^2 + y^2 = s^2 - 2p$

Et ſumma quadrat. $\frac{x^4}{y^2} + \frac{y^4}{x^2} = (a - s)^2 - 2p$.

Adeoque his additis $\frac{x^4}{y^2} + x^2$ &c. $= (a - s)^2 + s^2 - 4p = b$

Sive, ſubſtit. $\frac{s^3}{2s + a}$ pro $p$ * - - $(a - s)^2 + s^2 - \frac{4s^3}{2s + a} = b$

Sive, concinnando, - - - - $s^2 = -\frac{b}{a}s + \frac{1}{2}a^2 - \frac{1}{2}b$

Et per reductionem, - - - - $s = -\frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} + \frac{1}{2}a^2 - \frac{1}{2}b}$.

Invento $s$, innoteſcit $p = \frac{s^3}{2s + a}$. Itemque ob $s - y$ $(= x) = \frac{p}{y}$ prodit $sy - y^2 = p$, et $y = \frac{1}{2}s + \sqrt{\frac{1}{4}s^2 - p}$. Invento autem $y$ innoteſcit $x$, et proinde quatuor proportionales quæſiti.

---

* $\frac{s^3}{2s + a} = p$. Nam - - $\frac{x^2}{y} + \frac{y^2}{x} = a - s$.

Adeoque, - - - $x^3 + y^3 (= \overline{a - s}\, xy) = (a - s)\, p$

Itemque, per *Prob.* II. $x^3 + y^3 = s^3 - 3sp$.

Adeoque, - - - - - - - - $s^3 - 3sp = ap - sp$

Et - - - - - - - - - - - $s^3 = 2sp + ap$.

Ergo, - - - - - - - - $\frac{s^3}{2s + a} = p$.

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In hac ſolutione ſecuti ſumus *Simpſon*. Eundem valorem $s$, at operoſius quidem, contra morem ſuum, exquirit *Newtonus*. Aliam quoque et elegantem hujus problematis (necnon inſequentis) ſolutionem a *De Moivre* inventam proferre jam libet. Vide *Sanderſon*. *Algebra*, Vol. 1, p. 263.

86. Deſignantibus $x^3$, $x^2y$, $xy^2$, $y^3$ quatuor proportionales quæſitos, erit

$x^3 + x^2y + xy^2 + y^3 = a$. Adeoque, $(x+y)(x^2+y^2) = a$.
$x^6 + x^4y^2 + x^2y^4 + y^6 = b$. Adeoque, $(x^2+y^2)(x^4+y^4) = b$.

Abbreviandi gratiâ, finge $x + y = z$, et $x^2 + y^2 = v$.

Eritque - - - - - - $zv = a$, et $v = \frac{a}{z}$

Itemque - - $v \times (x^4 + y^4) = b$, et $x^4 + y^4 = \frac{b}{v}$.

Cæterum ob $x + y = z$, erit $x^2 + y^2 + 2xy = z^2$
Sive, ſubſtituendo, - - - $v + 2xy = z^2$
Adeoque, - - - - - - - $2xy = z^2 - v$.
Et quadrando, - - - - - $4x^2y^2 = z^4 - 2vz^2 + v^2$.
Porro, ob $x^2 + y^2 = v$, erit $x^4 + y^4 + 2x^2y^2 = v^2$
Sive, ſubſtituendo, - - - $\frac{b}{v} + 2x^2y^2 = v^2$

Adeoque, - - - - - - - $2x^2y^2 = v^2 - \frac{b}{v}$

Et, - - - - - - - - $4x^2y^2 = 2v^2 - \frac{2b}{v}$

Erit igitur, - $z^4 - 2vz^2 + v^2 (= 4x^2y^2) = 2v^2 - \frac{2b}{v}$

Adeoque, - - - $z^4 - 2vz^2 = v^2 - \frac{2b}{v}$

Sive, ob $v = \frac{a}{z}$, - - $z^4 - 2az = \frac{a^2}{z^2} - \frac{2bz}{a}$

Adeoque, - - - $z^6 + \left(\frac{2b}{a} - 2a\right)z^3 = a^2$

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Unde per reductionem § 48, elicietur $z$ et exinde $v$. His autem cognitis innoteſcent $x$ et $y$. Erat enim,

$$x^2+y^2+2xy=z^2$$

Subduc - - - $x^2+y^2 \quad = v$

Eritque - - - * $\quad 2xy=z^2-v$

Atque harum differentia $x^2+y^2-2xy=v-(z^2-v)=2v-z^2$

Et extracta radice - $x \quad -y \quad =\sqrt{2v-z^2}$

Huic adde - - - $x \quad +y \quad =z$

Eritque - - - $2x$ * $\quad =z+\sqrt{2v-z^2}$

Et - - - $x \quad =\frac{1}{2}z+\frac{1}{2}\sqrt{2v-z^2}$

Erit quoque - - - $2y$ * $\quad =z-\sqrt{2v-z^2}$

Et - - - - - - $y \quad =\frac{1}{2}z-\frac{1}{2}\sqrt{2v-z^2}$

Cognitis igitur $x$ et $y$, innoteſcent quatuor proportionales quæſiti.

Exemp. Sit $a=30$, et $b=340$; erat, per *præced.*

$$z^6=\frac{2a^2-2b}{a}z^3+a^2$$

Sive, poſito $\frac{a^2-b}{a}=d$, $z^6=2dz^3+a^2$

Adeoque - - - $z^3=d+\sqrt{d^2+a^2}$

Cæterum $d=\frac{a^2-b}{a}=\frac{900-340}{30}=\frac{56}{3}$

Adeoque $d^2+a^2=\frac{3136}{9}+900=\frac{11236}{9}$

Et - $\sqrt{d^2+a^2}=\frac{106}{3}$.

Adeoq. $z^3=d+\sqrt{d^2+a^2}=54=27\times 2$. Sit $c^3=2$,

Eritq. $z^3=27c^3$, et proinde $z=3c$. Sed $v=\frac{a}{z}=\frac{30}{3c}=\frac{10}{c}=\frac{10c^2}{c^3}=5c^2$.

Itemque $2v-z^2=10c^2-9c^2=c^2$, et $\sqrt{2v-z^2}=c$.

Ergo $x=\frac{z+\sqrt{2v-z^2}}{2}=\frac{3c+c}{2}=2c$. Et $y=\frac{z-\sqrt{2v-z^2}}{2}=c$.

| | | | | |
|---|---|---|---|---|
| Adeoque proportionales | $x^3$, | $x^2y$, | $xy^2$, | $y^3$ |
| Evadent - - - - - | $8c^3$, | $4c^3$, | $2c^3$, | $1c^3$ |
| Sive - - - - - - | 16, | 8, | 4, | 2. |

## PROB. VI.

87. *Invenire quinque numeros continue proportionales, quorum datur ſumma* a, *et quadratorum ſumma* b.

Deſignantibus $x$, $y$ & $z$ medios numeros quæſitos, deſignabunt $\frac{x^2}{y}$ et $\frac{z^2}{y}$ extremos. Habentur itaque æquationes $\frac{x^2}{y} + x + y + z + \frac{z^2}{y} = a$, et $\frac{x^4}{y^2} + x^2 + y^2 + z^2 + \frac{z^4}{y^2} = b$.

Ponamus $x + z = u$, eritque $\frac{x^2}{y} + \frac{z^2}{y} = a - u - y$; et propter proportionalitatem, tam $\frac{x^2z^2}{y^2}$ quam $xz = y^2$.

Adeoque per *Prob.* II. $\frac{x^4}{y^2} + \frac{z^4}{y^2} = (a - u - y)^2 - 2y^2$.

Et ſimiliter, $x^2 + z^2 = u^2 - 2y^2$. Seu dividendo per $y$, $\frac{x^2}{y} + \frac{z^2}{y} = \frac{u^2 - 2y^2}{y}$. Adeoque, ſubſtituendo erit

Summa proport. - - - $\frac{u^2 - 2y^2}{y} : + u : + y = a$

Et ſum. quad. $(a - u - y)^2 - 2y^2 : + u^2 - 2y^2 : + y^2 = b$

Quibus reductis, erit

$$a^2 - 2au - 2ay + 2u^2 + 2uy - 2y^2 = b$$

Et - - - - - - - $ay - u^2 - uy + y^2 = 0$

Harum priori addatur duplum poſterioris,

Eritque - - - $a^2 - 2au = b$; ſive $u = \frac{1}{2}a - \frac{\frac{1}{2}b}{a}$

Cæterum tranſponendo, fuit $y^2 + ay - uy = u^2$

Adeoque - - - - - $y = \frac{u - a}{2} + \sqrt{\frac{(u - a)^2}{4} + u^2}$.

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Cognitis autem $u$ et $y$, innoteſcent $x$ et $z$, et proinde quinque proportionales quæſiti.

*Aliter.*

88. Habitis jam ſecundum notationem *Moivreanam*, æquationibus

$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = a$, adeoque $\frac{x^5 - y^5}{x - y} = a$

$x^8 + x^6y^2 + x^4y^4 + x^2y^6 + y^8 = b$, adeoque $\frac{x^{10} - y^{10}}{x^2 - y^2} = b$.

Dividendo quartam æquationem per tertiam, erit

$\frac{x^5 + y^5}{x + y}$ ſeu $x^4 - x^3y + x^2y^2 - xy^3 + y^4 = \frac{b}{a}$

Adde - $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = a$

Eritque - - $2x^4 + 2x^2y^2 + 2y^4 = a + \frac{b}{a}$

Adeoque $x^4 + x^2y^2 + y^4 = \frac{1}{2}a + \frac{\frac{1}{2}b}{a} =$ puta $s$.

Subduc $x^4 - x^3y$, &c.
ab $x^4 + x^3y$, &c. eritque, $2x^3y + 2xy^3 = a - \frac{b}{a}$

Adeoque - - - - $x^3y + xy^3 = \frac{1}{2}a - \frac{\frac{1}{2}b}{a} =$ puta $d$.

Sive - - - - - - - $(x^2 + y^2)\, xy = d$.

Sive, poſitis $xy = z$,
Et $x^2 + y^2 = v$, - - - - - $zv = d$, adeoq. $z = \frac{d}{v}$.

Cæterum erat - - - $x^4 + x^2y^2 + y^4 = s$

Et $x^2 + y^2 = v$, quadr. dat, $x^4 + 2x^2y^2 + y^4 = v^2$

Et ſubducendo - - - - - $x^2y^2 = v^2 - s$

Adeoque - - $\frac{d^2}{v^2}\ (= z^2 = x^2y^2) = v^2 - s$

Sive - - $v^4 = sv^2 + d^2$.

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Unde elicietur $v$ et exinde $z$. Hiſce vero inventis innoteſcent $x$ et $y$.

Nam ob $xy = z$, erit - . $2xy = 2z$

Cæterum fuit - $x^2 + y^2 = v$

Et his additis, $x^2 + 2xy + y^2 = v + 2z$

Et radice extracta, $x + y = \sqrt{v + 2z}$

Erit quoque - $x - y = \sqrt{v - 2z}$

Et his additis, - $2x = \sqrt{v + 2z} + \sqrt{v - 2z}$

Sive - - $x = \dfrac{\sqrt{v + 2z} + \sqrt{v - 2z}}{2}$

Et - - - $y = \dfrac{\sqrt{v + 2z} - \sqrt{v - 2z}}{2}$

Exemp. Sit $a = 62$, $b = 1364$. Erit $s = \frac{1}{2} a + \frac{\frac{1}{2}b}{a} = 42$, et $d = \frac{1}{2} a - \frac{\frac{1}{2}b}{a} = 20$. Adeoque $v^2 = \frac{1}{2} s + \sqrt{\frac{1}{4}s^2 + d^2} = 50 = 25 \times 2$. Sit $c^2 = 2$, eritque $v^2 = 25c^2$, et $v = 5c$. Adeoque $\sqrt{v + 2z} = \sqrt{5c + 4c} = \sqrt{9c} = 3\sqrt{c}$. Et $\sqrt{v - 2z} = \sqrt{c}$.

Erit igitur $x = \dfrac{\sqrt{v + 2z} + \sqrt{v - 2z}}{2} = \dfrac{3\sqrt{c} + \sqrt{c}}{2} = 2\sqrt{c}$

Et - $y = \sqrt{c}$.

Et proinde $x^2 = 4c$, $xy = 2c$, $y^2 = c$.

Et proportionales $x^4$, $x^3y$, $x^2y^2$, $xy^3$, $y^4$

Evadent $16c^2$, $8c^2$, $4c^2$, $2c^2$, $1c^2$

Sive, - 32, 16, 8, 4, 2.

89. His conſtare poteſt quanta ſit in ſolutionum inventione varietas; et obiter, quod alii modi ſint aliis multo concinniores. Quapropter, ſi in primas de ſolutione problematis alicujus cogitationes modus computationi male accommodatus inciderit, relationes quantitatum iterum evolvendæ ſunt, donec modum quam poteris idoneum et elegantem machinatus fueris.

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## De exterminatione plurium quantitatum incognitarum e pluribus æquationibus.

90. Si duæ quantitates incognitæ è tribus æquationibus tollendæ ſunt, aut plures è pluribus; conferendo inter ſe harum quaſque binas, per vices, exterminandi opus per regulas jam ante traditas, gradatim peragetur.

91. Ex tribus pluribuſve æquationibus inter ſe collatis erui quoque poſſint *formulæ*, ad inſtar præcedentium, pro exterminanda $x$, ubi ſit unius, duarum, pluriumve dimenſionum. Cæterum hæ formulæ ad gradus plane enormes aſſurgunt, ubi altiores dimenſiones $x$ tollendæ ſunt. Sic exterminato $x$ ex tribus quadraticis, prodit æquatio $16^{mi}$ gradus; ex tribus cubicis, $81^{mi}$ gradus; ex tribus biquadraticis, $256^{mi}$ gradus; et ſic deinceps. Nec huic incommodo ſatis medetur methodus exterminationis quam invenit *Bezout*, (*ſupra cit.* § 75) quippe quæ in caſibus jam memoratis, ad æquationes $12^{mi}$, $49^{mi}$, et $156^{mi}$ gradus reſpective, deducit.

Solutiones problematum quorundam plures quantitates incognitas parientium afferre jam libet.

### PROB. I.

92. *Diſſimiles duarum pluriumve rerum miſturas ita componere, ut res illæ commiſtæ datam inter ſe rationem acquirant.*

Deſignent A, B, C, &c. res miſtas, et $d$, $e$, $f$, $g$, $h$, &c. proportiones earundem in miſturis. Sit unius miſturæ data quantitas $d\mathrm{A} + e\mathrm{B} + f\mathrm{C}$, alterius eadem quantitas $g\mathrm{A} + h\mathrm{B} + k\mathrm{C}$, et eadem tertiæ $l\mathrm{A} + m\mathrm{B} + n\mathrm{C}$. Sintque

hæ miſturæ ita componendæ ut compoſitionis eadem quantitas contineat $pA + qB + rC$. Finge $x$, $y$, et $z$, partes eſſe trium miſturarum, ſive numeros eſſe per quos ſi tres datæ miſturæ reſpective multiplicentur earum aggregatum evadet $pA + qB + rC$.

Eſt itaque $\left.\begin{array}{l} dxA + exB + fxC \\ + gyA + hyB + kyC \\ + lzA + mzB + nzC \end{array}\right\} = pA + qB + rC.$

Adeoque, collatis terminis, ob $dxA + gyA + lzA + = pA$, atque ita deinceps,

Erunt $\left\{\begin{array}{ll} dx + gy + lz = p, & \text{adeoque } x = \frac{p - gy - lz}{d} \\ ex + hy + mz = q, & \text{adeoque } x = \frac{q - hy - mz}{e} \\ fx + ky + nz = r, & \text{adeoque } x = \frac{r - ky - nz}{f}. \end{array}\right.$

Et rurſus ſunt $\left\{\begin{array}{l} \frac{p - gy - lz}{d} = \frac{q - hy - mz}{e}, \text{ adeoq. } y = \frac{ep - dq + dmz - elz}{eg - dh} \\ \frac{q - hy - mz}{e} = \frac{r - ky - nz}{f}, \text{ adeoq. } y = \frac{fq - er + enz - fmz}{fh - ek} \end{array}\right.$

Et denique, ſi abbrevientur hi valores $y$, ſcribendo $\alpha$ pro $ep - dq$, $\beta$ pro $dm - el$, $\gamma$ pro $eg - dh$, $\delta$ pro $fq - er$, $\zeta$ pro $en - fm$, et $\theta$ pro $fh - ek$; evadet $\frac{\alpha + \beta z}{\gamma} = \frac{\delta + \zeta z}{\theta}$, & per reductionem, $z = \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta}$. Invento $z$, pone $\frac{\alpha + \beta z}{\gamma} = y$, et $\frac{p - gy - lz}{d} = x$.

Exemp. Si tres ſint metallorum colliquefactorum miſturæ, quarum primæ pondo continet argenti uncias 12,

æris 1, & ſtanni 3, ſecundæ pondo continet argenti unciam 1, æris 12, et ſtanni 3, et tertiæ pondo continet æris uncias 14, ſtanni 2, et argenti nihil; ſintque hæ miſturæ ita componendæ ut pondo compoſitionis contineat argenti uncias 4, æris 9, et ſtanni 3: Pro $d, e, f; g, h, k; l, m, n; p, q, r$, ſcribe 12, 1, 3; 1, 12, 3; 0, 14, 2; 4, 9, 3 reſpective, & erit $\alpha$ $(= ep - dq = 1 \times 4 - 12 \times 9) = -104$, & $\beta$ $(= dm - el = 12 \times 14 - 1 \times 0) = 168$, & ſic $\gamma = -143$, $\delta = 24$, $\zeta = -40$, & $\theta = 33$. Adeoque $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{-3432 + 3432}{5720 - 5544}\right) = 0$,

$y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-104 + 0}{-143}\right) = \frac{8}{11}$, & $x \left(= \frac{p - gy - lz}{d} = \frac{4 - \frac{8}{11}}{12}\right) = \frac{3}{11}$. Quare ſi miſceantur $\frac{8}{11}$ partes pondo miſturæ ſecundæ, $\frac{3}{11}$ partes pondo primæ, et nihil tertiæ, aggregatum erit pondo continens quatuor uncias argenti, novem æris, et tres ſtanni. Uti ex eo conſtat, quod $\frac{3}{11}$ partes pondo primæ $= \frac{3}{11} \times \left(\frac{12 + 1 + 3}{16}\right) = \frac{36 + 3 + 9}{176}$. Et $\frac{8}{11}$ partes ſecundæ $= \frac{8}{11} \times \left(\frac{1 + 12 + 3}{16}\right) = \frac{8 + 96 + 24}{176}$, et proinde aggregatum $= \frac{44 + 99 + 33}{176} = \frac{4 + 9 + 3}{16}$.

93. Cæterum notandum eſt quod diſſimiles hujuſmodi miſturas ſecundum datam quamlibet rationem componere aliquando impoſſibile evadet, quando, ſcil. valores $x$, $y$ vel $z$ prodeunt negativi; tales enim in quæſtionis ſolutione valores affirmativos tantum poſtulantis, locum habere nequeunt. Ut ſi primæ pondo contineat argenti, æris et

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ſtanni 3, 6, et 7 uncias reſpective, ſecundæ pondo, 1, 12, et 3 uncias, tertiæ pondo 4, 2 et 10 uncias, et deſideretur compoſitio cujus pondo contineat 4, 9 et 3 uncias: Pro $d$, $e$, $f$, &c. ſcriptis 3, 6, 7, &c. erit $\alpha = -3$, $\beta = -18$, $\gamma = -30$, $\delta = 45$, $\zeta = 46$, et $\theta = 66$. Adeoque $z = \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{-198 + 1350}{-1380 + 1188} = \frac{1152}{-192} = -6$, $y = \frac{\alpha + \beta z}{\gamma} = -\frac{7}{2}$, & $x = \frac{p - gy - lz}{d} = \frac{4 + \frac{7}{2} + 24}{3} = \frac{63}{6} = \frac{21}{2}$. Reſpiciunt nimirum valores $y$ et $z$ jam negativi ad quæſtionis caſum in praxi impoſſibilem, nempe *Diſſimiles rerum miſturas ita ab invicem ſecernere, ut reſiduum res illas datâ quadam ratione inter ſe commiſtas exhibeat.* Sic in hoc exemplo, ſi a $10\frac{1}{2}$ pondo miſturæ primæ demantur $3\frac{1}{2}$ pondo ſecundæ et 6 pondo tertiæ, reſiduum erit pondo continens argenti æris et ſtanni 4, 9 & 3 uncias reſpective.

## PROB. II.

94. *Invenire valores* x, y *et* z *in æquationibus* $x^2 + xy = 10$, $y^2 + yz = 21$, *et* $z^2 + zx = 24$.

Subſtituatur $nx$ pro $y$, et $mx$ pro $z$, eritque

$$\left.\begin{aligned} x^2 + nx^2 &= 10, \\ n^2x^2 + nmx^2 &= 21, \\ m^2x^2 + mx^2 &= 24, \end{aligned}\right\} \text{ ſive } \left\{\begin{aligned} x^2 &= \frac{10}{1+n} \\ x^2 &= \frac{21}{n^2+nm} \\ x^2 &= \frac{24}{m^2+m}. \end{aligned}\right.$$

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Adeoque $\frac{24}{m^2+m}\,(=x^2)=\frac{10}{1+n}$, ergo $n=\frac{5m.\overline{m+1}-12}{12}$.

Itemq. $\frac{24}{m^2+m}\,(=x^2)=\frac{21}{n^2+nm}$, ergo $8n^2+8nm=7m.\overline{m+1}$.

Sed - - $n^2=\frac{25m^2.\overline{m+1}^2-120m.\overline{m+1}+144}{144}$

Et - - $nm=\frac{5m^2.\overline{m+1}-12m}{12}$.

Adeoq. $7m.\overline{m+1}\,(=8n^2+8nm)=\frac{25m^2.\overline{m+1}^2-120m.\overline{m+1}+144}{18}$

$$+\frac{10m^2.\overline{m+1}-24m}{3}.$$

Sive concinnando, $25m^4+110m^3-161m^2-390m+144=0$.
Et divid. per $m-2$, $25m^3+160m^2+159m-72=0$.
Unde per reſolutionem, prodit $m=2$.

Adeoque $n\left(=\frac{10.3-12}{12}\right)=\frac{3}{2}$, $x\left(=\sqrt{\frac{10}{1+1\frac{1}{2}}}\right)=2$, $y=3$, et $z=4$.

## PROB. III.

95. *Si boves* a *depaſcant pratum* b *in tempore* c; *& boves* d *depaſcant pratum æque bonum* e *in tempore* f, *& gramen uniformiter creſcat: Quæritur quot boves depaſcent pratum ſimile* g, *in tempore* h.

Diſtinguantur boves cujuſque prati binos in claſſes; ſint boves $a=r+s$, quorum boves $r$ depaſcant gramen quod in prato $b$ ante boves admiſſos creverat, boves autem $s$ graminis incrementum poſt. Similiter, ſint boves $d=t+u$. Et boves quæſiti $x=y+z$. Sed boves prioris claſſis $r$, $t$, et $y$, dato tempore, ſunt ut prata depaſcenda, (quippe quæ per hypotheſin æque bona ſunt) et datis pratis, reciproce ut

depaſcendi tempora, et proinde neutris datis, ut prata directe et depaſcendi tempora reciproce, hoc eſt, ut $\frac{b}{c}$, $\frac{e}{f}$, $\frac{g}{h}$ reſpective. Adeoque per analogiam, $r : t :: \frac{b}{c} : \frac{e}{f}$, et ductis mediis et extremis $\frac{tb}{c} = \frac{cr}{f}$, adeoque $t = \frac{cer}{bf}$. Et ſimiliter, $y = \frac{cgr}{bh}$. Sunt quoque boves poſterioris claſſis $s$, $u$, $z$ & tempora depaſcendi conjunctim, ut prata $b$, $e$, $g$ et tempora graminis in hiſce creſcendi (uniformiter per hypotheſin) conjunctim; ſive, æquipollentibus his temporibus, boves $s$, $u$, $z$ ut prata $b$, $e$, $g$ reſpective. Adeoque, per analogiam, boves $u = \frac{es}{b}$, et $z = \frac{gs}{b}$. Subſtituantur in æquationem ſecundam et tertiam valores $t$, $y$, $u$, et $z$ jam inventi, et prodibunt

$$a = r + s, \text{ adeoque } r = a - s.$$

$$d = \frac{cer}{bf} + \frac{es}{b} = \frac{ce \times \overline{a - s}}{bf} + \frac{es}{b}, \text{ adeoque } s = \frac{bdf - cea}{ef - ce}.$$

$$x = \frac{cgr}{bh} + \frac{gs}{b} = \frac{cg \times \overline{a - s}}{bh} + \frac{gs}{b}, \text{ adeoque } s = \frac{bhx}{gh} - \frac{cga}{cg}.$$

$$\frac{bhx - cga}{gh - cg} (= s) = \frac{bdf - cea}{ef - ce}, \text{ adeoque}$$

$$befhx - bcehx = bdfgh - acegh - bcdfg + acefg$$

$$\text{Sive} \quad - \quad - \quad x = \frac{bdfgh - acegh - bcdfg + acefg}{befh - bceh}.$$

96. Longe elegantius, at ſubtilius quidem, ſolutionem expedit *Newtonus*. Primo enim, exquirit quot boves in tempore $b$ depaſcerent graminis incrementum in prato $e$ per tempus $c$; deinde, quot boves in eodem tempore $b$ depaſcerent graminis incrementum in eodem prato $e$ per tempus

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$b - c$; nam ex horum ſumma conflatur totus numerus boum quibus per tempus $b$ paſcendis ſufficeret totum graminis incrementum in hoc prato $e$ per tempus $b$. Et inde per analogiam, colligit quot boves depaſcerent pratum $g$ in eodem tempore $b$. Hoc præmiſſo, ipſius ſolutionem, notulis quibuſdam ſubjectis proferre jam libet.

Si boves $a$ in tempore $c$ depaſcant pratum $b$; tum per analogiam, boves $\frac{e}{b}a$ in eodem tempore $c$ (a), vel boves $\frac{ec}{bf}a$ in tempore $f$ (b), vel boves $\frac{ec}{bb}a$ in tempore $b$, depaſcent pratum $e$: puta ſi gramen poſt tempus $c$ non creſceret. Sed cum propter graminis incrementum boves $d$ in tempore $f$ depaſcant ſolummodo pratum $e$, ideo graminis in prato $e$ incrementum illud per tempus $f - c$ tantum erit quantum per ſe ſufficit paſcendis bobus $d - \frac{eca}{bf}$ per tempus $f$, hoc eſt quantum ſufficit paſcendis bobus $\frac{df}{b} - \frac{eca}{bb}$ per tempus $b$ (c). Et in tempore $b - c$ per analogiam

(a) $b : e :: a : \frac{ea}{b}$ ob boves ut prata, dato graminis incremento per tempus $c$.

(b) $\left.\begin{array}{l} \frac{1}{c} : \frac{1}{f} :: \frac{ea}{b} : \frac{cea}{bf} \\ \frac{1}{c} : \frac{1}{b} :: \frac{ea}{b} : \frac{cea}{bb} \end{array}\right\}$ ob boves reciproce ut tempora depaſcendi pratum datum $e$, dato graminis incremento per tempus $c$.

(c) $\frac{1}{f} : \frac{1}{b} :: d - \frac{cea}{bf} : \frac{fd}{b} - \frac{fcea}{bfb}$ ſeu $\frac{fd}{b} - \frac{cea}{bb}$ ob boves dato graminis incremento per tempus $f - c$, reciproce ut depaſcendi tempora.

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tantum erit incrementum quantum per ſe ſufficit paſcendis bobus $\frac{h-c}{f-c}$ in $\frac{df}{h} - \frac{eca}{bh}$, ſive $\frac{bdfh - ecah - bdcf + acc^2}{bfh - bch}$ (d) [per tempus $h$.] Hoc incrementum adjice bobus $\frac{acc}{bh}$ et prodibit $\frac{bdfh - ecah - bdcf + ecfa}{bfh - bch}$ numerus boum quibus paſcendis ſufficit pratum $e$ per tempus $h$. Adeoque per analogiam, pratum $g$ bobus $\frac{bdfgh - ecagh - bdcgf + ecfga}{befh - bceh}$ per idem tempus $h$ paſcendis ſufficiet.

Exemp. Si 12 boves depaſcant $3\frac{1}{3}$ jugera prati in 4 ſeptimanis; & 21 boves depaſcant 10 jugera conſimilis prati in 9 ſeptimanis; quæritur quot boves depaſcant 24 jugera in 18 ſeptimanis? Reſp. 36. Iſte enim numerus invenietur ſubſtituendo in $\frac{bdfgh - ecagh - bcdgf + ecfga}{befh - bceh}$ numeros 12, $3\frac{1}{3}$, 4, 21, 10, 9, 24, et 18, pro literis $a$, $b$, $c$, $d$, $e$, $f$, $g$ et $h$ reſpective.

Cæterum ſolutio forte haud minus expedita erit ſi è primis principiis ad formam ſolutionis præcedentis literalis eruatur. Utpote ſi 12 boves in 4 ſeptimanis depaſcant $3\frac{1}{3}$ jugera, tum per analogiam 36 boves in 4 ſeptimanis vel 16 boves in 9 ſeptimanis vel 8 boves in 18 ſeptimanis depaſcent 10 jugera: puta ſi gramen non creſceret. Sed

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(d) $f - c : h - c :: \frac{df}{h} - \frac{eca}{bh}$ ſeu $\frac{bdf - eca}{bh} : \frac{bdfh - ecah - bdcf + eac^2}{bfh - bch}$ ob boves ut graminis incrementa in prato $e$ per tempora $f - c$ et $h - c$.

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cum propter graminis incrementum 21 boves in 9 ſeptimanis depaſcant ſolummodo 10 jugera, illud graminis in 10 jugeris per poſteriores 5 ſeptimanis incrementum tantum erit quantum per ſe ſufficit exceſſui boum 21 ſupra 16, hoc eſt 5 bobus per 9 ſeptimanas, vel quod perinde eſt $\frac{5}{2}$ bobus per 18 ſeptimanas paſcendis. Et in 14 ſeptimanis (exceſſu 18 ſupra 4 primas) incrementum illud per analogiam tantum erit quantum ſufficiat 7 bobus per 18 ſeptimanas paſcendis; eſt enim 5 ſept. ad 14 ſeptimanas ut $\frac{5}{2}$ boves ad 7 boves. Quare 8 bobus quos 10 jugera ſine incremento graminis paſcere poſſunt per 18 ſeptimanas adde hoſce 7 boves quibus paſcendis ſolum incrementum graminis ſufficit, et ſumma erit 15 boves. Ac denique ſi 10 jugera 15 bobus per 18 ſeptimanas paſcendis ſufficiant, tum per analogiam 24 jugera per idem tempus ſufficient 36 bobus.

In priore ſolutione literali ſeptem adſciſcuntur quantitates incognitæ, in poſteriore ne una quidem. Luculenter profectò evincit hoc exemplum non ſemper ad Algebram neceſſario recurrendum eſſe in problematum ſolutionem quæ primâ fronte difficillima videantur. Verùm *Newtoni* ſagacitas haud cuique Analyſtæ contingit.

## De exterminatione quantitatum surdarum ex æquationibus.

97. Tollendæ ſunt ſurdæ quadraticæ ex æquatione qualibet haud plures quam quatuor terminos complectente, ſive hi omnes ſint irrationales ſive non, per involutionem iteratam, ope Regulæ VI, § 25. Sic æquatio $\sqrt{a} - \sqrt{b} - \sqrt{c} - \sqrt{d} = 0$, (ubi $\sqrt{a}$, $\sqrt{b}$, &c. quantitates quaſcun-

que ſurdas deſignant ſive ſimplices ſive complexas) tranſponendo, dat $\sqrt{a} - \sqrt{b} = \sqrt{c} + \sqrt{d}$; et quadrando, $a - 2\sqrt{ab} + b = c + 2\sqrt{cd} + d$; atque iterum tranſponendo, $2\sqrt{ab} + 2\sqrt{cd} = a + b - c - d =$ puta $g$; & partibus iterum quadratis, $4ab + 8\sqrt{abcd} + 4cd = g^2$; atque denuo tranſponendo, $\sqrt{abcd} = \frac{g^2 - 4ab - 4cd}{8} =$ puta $h$; partibus autem denuo quadratis emergit æquatio quatuor dimenſionum a ſurdis libera, $abcd = h^2$. Verùm ſi datæ æquationi ſubquadraticæ accedat terminus quintus, ſive hic ſit rationalis ſive non, amovendæ ex æquatione aſymmetriæ impar erit hæc regula utcunque diſtribuas terminos. Nam ex unâ æquationis parte neceſſario ſiſtentur tres, quatuor, aut quinque termini, adeoque ex alterâ duo, unus, aut nullus, reſpective; at ſurda quantitas binomia quadrando parit unum terminum irrationalem, (ut in exemplo allato) trinomia parit tres, quadrinomia ſex, et quinquinomia decem: ergo ex involutione partium quotieſcunque iteratâ provenient termini ſurdi numero plures quam quatuor, vel ſaltem non pauciores, in iſto caſu ubi quatuor terminis ſurdis accedit quintus terminus qui ſit rationalis. Minus itaque valet hæc regula ad ſurdarum quadraticarum in omni caſu exterminationem, licet harum genus ſit ſimpliciſſimum. Et *a fortiori*, intra limites adhuc arctiores concludetur ſurdarum cubicarum exterminatio.

98. Cæterum quantitates *quotcunque* ſurdas ex æquationibus tollere licet ope methodi *Fermatianæ*, eas nempe fingendo literis quibuſlibet æquales; deinde, ex æquationibus reſultantibus tollendo gradatim has literas, per regulas præcedentes, emerget tandem æquatio libera ab omni aſymmetria. Quemadmodum ſi ſit $\sqrt{ay} - \sqrt{a^2 - ay} = 2a + \sqrt[3]{ay^2}$, ſcribendo $t$ pro $\sqrt{ay}$, $v$ pro $\sqrt{a^2 - ay}$, et $x$ pro $\sqrt[3]{ay^2}$, ha-

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bentur quatuor æquationes $t - v = 2a + x$, $t^2 = ay$, $v^2 = a^2 - ay$, et $x^3 = ay^2$. Sed per primam æquationem,

Erit - - - - - - - - $2a + x + v = t$

Et - $4a^2 + 4ax + 4av + x^2 + 2xv + v^2 = (t^2 =)\ ay$

Adeoque, ſubſtituendo $a^2 - ay$ pro $v^2$, et tranſponendo,

$$x^2 + 4ax - 2ay + 5a^2 = -2xv - 4av$$

Adeoque - - - $\dfrac{x^2 + 4ax - 2ay + 5a^2}{2x + 4a} = -v.$

Et $\dfrac{x^4 + 8ax^3 - 4ayx^2 + 26a^2x^2 - 16a^2yx + 40a^3x + 4a^2y^2 - 20a^3y + 25a^4}{4x^2 + 16ax + 16a^2}$

$= (v^2 =)\ a^2 - ay$. Unde ſubſtituendo $ay^2x$ pro $x^4$, $8a^2y^2$ pro $8ax^3$, et concinnando, evadit

$$22ax^2 + y^2x + 24a^2x + 12ay^2 - 4a^2y + 9a^3 = 0.$$

Sive, abbreviandi gratiâ ſcribendo

A pro $22a$, B pro $y^2 + 24a^2$, et C pro $12ay^2 - 4a^2y + 9a^3$,

Erit - - - - - - $Ax^2 + Bx + C = 0$

Adeoque - - - - $Ax^3 + Bx^2 + Cx = 0$

Sed $x^3 - ay^2 = 0$, $\therefore$ $Ax^3 - Aay^2 = 0$

$$* \ Bx^2 + Cx + Aay^2 = 0.$$

$ABx^2 + B^2x + BC = 0$ $\qquad$ $A^2ay^2x^2 + ABay^2x + ACay^2 = 0$

$ABx^2 + ACx + A^2ay^2 = 0$ $\qquad$ $BCx^2 + C^2x + ACay^2 = 0$

$* \ (B^2 - AC)x + BC - A^2ay^2 = 0$ $\qquad$ $(BC - A^2ay^2)\,x + C^2 - ABay^2 \ * = 0$

$$(B^2 - AC)(BC - A^2ay^2)\,x + (BC - A^2ay^2)^2 = 0$$

$$(B^2 - AC)(BC - A^2ay^2)\,x + (B^2 - AC)(C^2 - ABay^2) = 0$$

$$* \ (BC - A^2ay^2)^2 - (B^2 - AC)(C^2 - ABay^2) = 0$$

$$\text{Sive} \quad A^3a^2y^4 + B^3ay^2 + C^3 - 3ABCay^2 = 0.$$

Unde ſubſtituendo valores A, B, et C, et dividendo per $a$, prodit æquatio a ſurdis expurgata

$$x^8 + 1008a^2y^6 - 1464a^3y^5 - 2762a^4y^4 + 3680a^5y^3 + 2916a^6y^2 - 972a^7y + 729a^8 = 0.$$

## DE PROBLEMATIS INDETERMINATIS.

99. Hactenus prolatæ ſunt quæſtiones quæ tot pariunt æquationes quot habent quantitates incognitas, et quæ idcirco, ad ſolutionem determinatam ſeu certam perducunt. Quæſtiones jam apponemus quæ conditionum inopia laborantes, pauciores præbent æquationes quam quantitates incognitas, atque ſolutionem indeterminatam ſeu incertam ſortiuntur.

### P R O B. I.

100. *Datis et miſturæ et miſtorum gravitatibus ſpecificis, invenire proportionem miſtorum inter ſe.*

Sint $x$ et $y$ miſtorum moles, $a$ et $b$ horum gravitates ſpecificæ, & $e$ gravitas ſpecifica miſturæ, et cum gravitas abſoluta ſeu pondus componatur ex mole corporis et gravitate ſpecifica, erit $ax$ pondus prioris, $by$ pondus poſterioris, et $e \times \overline{x+y}$ pondus miſturæ; adeoque $ax + by = ex + ey$, et tranſponendo, $ax - ex = ey - by$, ſive $\overline{a-e} \times x = \overline{e-b} \times y$, indeque $x : y :: e - b : a - e$. Sed cognita jam proportione molium $x$ et $y$ inter ſe, et gravitatum ſpecificarum $a$ et $b$, componendo innoteſcet proportio ponderum miſtorum inter ſe.

Exempl. Sit $a$ auri gravitas experimentis hydroſtaticis comperta ut 19, $b$ argenti gravitas ut $10\frac{1}{3}$, et $e$ Coronæ *Hieronis* ut 17. Eritque moles auri in corona ad molem argenti, ut ($e - b : a - e :: 17 - 10\frac{1}{3} :: 19 - 17 :: 6\frac{2}{3} : 2 :: 20 : 6 ::$) 10 ad 3; adeoque, pondus auri in corona ad

pondus argenti, ut ($a \times \overline{e-b} : b \times \overline{a-e}$ :: $19 \times 10 : 10\frac{1}{3} \times 3$ ::) $190 : 31$; et pondus coronæ ad pondus argenti, ut $190 + 31 = 221$ ad $31$. Innoteſcit igitur proportio ponderum coronæ et miſtorum inter ſe. Cæterum quo eruantur ipſa miſtorum pondera, et quæſtio determinata ſit, poſtulatur altera conditio, nempe ut innoteſcat coronæ pondus, de hoc vero ſilet *Vitruvius*, *loc. cit.* § 12.

Si miſtura præter aurum et argentum, aliod quodvis metallum colliquefactum contineret, etiamſi daretur coronæ pondus, ipſa miſtorum pondera neutiquam detexiſſet *Archimedes*, ut ex ſeq. *prob.* patebit.

## PROB. II.

101. *Datis miſturæ ex tribus metallis conflatæ pondere, et miſtorum gravitatibus ſpecificis, invenire proportionem miſtorum inter ſe.*

Sint $x$, $y$, $z$, miſtorum puta auri, argenti et æris moles; $a$, $b$, $c$ horum gravitates ſpecificæ reſpective; miſturæ autem gravitas ſpecifica $g$, et pondus $p$: & inde duæ habentur æquationes $ax + by + cz = p$, et $ax + by + cz = g \times \overline{x + y + z}$; ex priore autem elicitur $x = \frac{p - by - cz}{a}$ & hoc valore $x$ in alteram ſubſtituto, concinnando prodit $y = \frac{agz - cgz + gp - ap}{bg - ag}$. Quoniam igitur valoribus $x$ et $y$ immiſcetur $z$, hoc incognito, illi determinari nequeunt.

102. Sufficiant hæc in ſpecimen ſolutionum problematum hujuſmodi indeterminatorum. Pergamus jam methodum generalem ſolvendi problemata de numeris integris & affirmativis docere, quam invenit ſummus analyſta *De Moivre.* Vide *Dodſon's Math. Repoſ. Vol.* 2.

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## PROB. I.

103. *Quæruntur valores integri et affirmativi* x *et* y *in æquatione* $9x - 7y = 13$.

Tranſp. erit $7y = 9x - 13$

Adeoque - $y = \frac{9x - 13}{7} = x - 1 + \frac{2x - 6}{7}$ (quod patet dividendo $9x - 13$ per 7). Si igitur ex hoc valore $y$, per quæſtionem integro, dematur maximus integer quem ſub ſe continet ſcilicet quotiens $x - 1$, erit reſiduum $\frac{2x - 6}{7}$ integer. Finge hoc æquari $a$, et inde erit $2x - 6 = 7a$, et $x = \frac{7a + 6}{2} = 3a + 3 + \frac{a}{2}$. Sed hic valor $x$ eſt quoque integer, ab hoc igitur dempto maximo integro $3a + 3$, erit reſiduum $\frac{a}{2}$ integer = puta $b$, et inde $a = 2b$. Comparato jam valore integro $a$ abſque reſiduo, regrediendo ſubſtituatur $2b$ pro $a$ in valorem præcedentem $x$, et prodibit $x = \frac{\overline{7 \times 2b} + 6}{2} = \frac{14b + 6}{2} = 7b + 3$,

et - $y = \frac{\overline{9 \times \overline{7b + 3}} - 13}{7} = \frac{\overline{9 \times 7b} + 14}{7} = 9b + 2$.

In hiſce autem valoribus $x$ et $y$, deſignare $b$ poteſt aut zero, aut integrum quemlibet majorem zero, ſic enim affirmativi exurgent horum valores, ſecus ſi minorem zero. Sit igitur

$b = 0$, erit $x = 0 + 3 = 3$, et $y = 0 + 2 = 2$
$b = 1$, $x = 7 + 3 = 10$, - $y = 9 + 2 = 11$
$b = 2$, - $x = 14 + 3 = 17$, - $y = 18 + 2 = 20$,
&c. &c. &c.

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Hinc in hujuſmodi æquationibus $fx - gy = h$, innumeri prodibunt quantitatum incognitarum valores, creſcente $x$ per coefficientem $y$ et creſcente $y$ per coefficientem $x$, in infinitum.

## PROB. II.

104. *Valente nummo quodam anglico* 21 *ſolidos, gallico autem* 17 *ſolidos, quot utriuſque* 100 *libras ſterlingas perſolvent; ſive in genere, quæruntur valores integri et affirmativi* x *et* y *in æquatione* $21x + 17y = 100 \times 20 = 2000$.

$$21x + 17y = 2000$$

Tranſp. et divid. erit $y = \frac{2000 - 21x}{17} = 117 - x + \frac{11 - 4x}{17}$

Sit $a = \frac{11 - 4x}{17}$, adeoq. $x = \frac{11 - 17a}{4} = 2 - 4a + \frac{3 - a}{4}$

Sit $b = \frac{3 - a}{4}$, adeoq. $a = 3 - 4b$.

Subſtituatur jam hic valor integer $a$ reſiduo vacans, in valorem $x$, et evadet

$x = \frac{11 - 17 \times \overline{3 - 4b}}{4} = 17b - 10$, et $y = \frac{2000 - 21 \times \overline{17b - 10}}{17}$ $= 130 - 21b$.

Cæterum quo $17b - 10$ $(= x)$ evadat affirmativus, ſtatuendus eſt $b$ major nihilo, et quo $130 - 21b$ $(= y)$ evadat affirmativus, limitem $\frac{130}{21} = 6\frac{4}{21}$ haud tranſcendere debet. Sortietur igitur $b$ ſex valores integros, ſcil.

| | | | | | | |
|---|---|---|---|---|---|---|
| | $b =$ 1, | 2, | 3, | 4, | 5, | 6, |
| Adeoque | $x =$ 7, | 24, | 41, | 58, | 75, | 92, |
| | $y =$ 109, | 88, | 67, | 46, | 25, | 4. |

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Hinc in æquationibus formæ $fx + gy = b$, intra certos limites concludentur valores affirmativi $x$ et $y$, creſcente $x$ per coefficientem $y$ et decreſcente $y$ per coefficientem $x$.

## PROB. III.

105. *Quæruntur valores integri et affirmativi* x *et* y *in æquatione* $5x + 8y = 1989$.

Tranſponendo et dividendo erit $x = \frac{1989 - 8y}{5}$, at hic valor $x$ per quæſtionem debet eſſe integer, unde pergendo ſecundum methodum *Moivreanam* in præcedentibus adhibitam, erui poteſt ſolutio.

106. Citius autem methodo inſequenti integrum quem quærimus reſiduo vacantem eruere licet, quam idcirco in poſterum uſurpabimus. Sumantur multipli denominatoris 5 terminis numeratoris 1989 et $8y$ proximi, ſive ſupra ſive infra fuerint, ſcilicet 1990 et $10y$, eritque $\frac{1990 - 10y}{5}$ integer, adeoque differentia quoque $\frac{1 - 2y}{5}$ integer, = puta $a$, et inde evadet $y = \frac{1 - 5a}{2}$. Similiter, ab hujus numeratore dematur multiplus proximus denominatoris 2, ſcil. $4a$, eritque $\frac{1 - a}{2}$ integer = puta $b$, unde evadet $a = 1 - 2b$.

Prodibit igitur
$$\begin{cases} y = \dfrac{1 - 5 \times \overline{1 - 2b}}{2} = 5b - 2, \\ \text{et } x = \dfrac{1989 - 8 \times \overline{5b - 2}}{5} = 401 - 8b. \end{cases}$$

Adeoque ob $5b - 2\ (= y)$ affirmativum, neceſſe eſt ut $b$ ſit major nihilo, at non tranſcendere debet limitem $\frac{401}{8} = 50\frac{1}{8}$, ſic enim negativus evaderet $401 - 8b\ (= x)$ Sortietur itaque $b$ valores integros 50, totidemque $x$ et $y$, ſcil.

$$\begin{array}{llllll} b = 1, & 2, & 3, & \&c. & 50, \\ x = 393, & 385, & 377, & \&c. & 1, \\ y = 3, & 8, & 13, & \&c. & 248. \end{array}$$

107. Cæterum notandum eſt in æquationibus formæ $fx \pm gy = b$, quo valores $x$ et $y$ emergant integri, neceſſe eſt ut coefficientes $f$ et $g$ ſint ad ſe invicem primi, aliter enim dividendo per maximum horum diviſorem communem, terminus abſolutus $b$ evadere poteſt fractionalis, ac idcirco ſolutio in numeris integris impoſſibilis. Sic habita æquatione $6x + 9y = 17$, dividendo per 3 erit $2x + 3y = 5\frac{2}{3}$.

*Aliquando contractius ſolvi poſſunt bujuſmodi æquationes.*

108. Sic propoſita æquatione $5x + 8y = 152$, ubi coefficientium alter ſcilicet 8 ſit diviſor termini abſoluti 152, erit

$$5x = (152 - 8y =)\ \overline{19 - y} \times 8$$

Adeoque $x : 19 - y :: 8 : 5$.

Erit igitur $x = 8,\ 16,\ 24,\ \&c.$

$19 - y = 5,\ 10,\ 15,\ \&c.$

Adeoque $y = 14,\ 9,\ 4.$

109. Propoſita æquatione $5x + 8y = 91 = 13 \times 7$, ubi ſumma coefficientium ſit termini abſoluti diviſor, manifeſtum eſt utramque quantitatem incognitam deſignari poſſe



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per quotum 7, et inde ex lege incrementi et decrementi prodeunt utriuſque valores,

$$\begin{cases} x = 7, & 15 \\ y = 7, & 2. \end{cases}$$

Aut propoſita æquatione $5x - 8y = 81 = 27 \times 3$, ubi differentia coefficientium ſit termini abſoluti diviſor, manifeſtum eſt $x$ et $y$ deſignari poſſe per quotum 27, et inde facile habentur reliqui utriuſque valores.

110. Propoſita æquatione $5x + 8y = 198$, ubi differentia coefficientium ſit termini abſoluti diviſor, finge $x + y = s$, adeoque $5x + 5y = 5s$ et $8x + 8y = 8s$. Sed differentiæ inter haſce et datam dabunt

$$\begin{cases} 3y = 198 - 5s, & \text{adeoque erit } s < \frac{198}{5} = 39\frac{3}{5} \\ 3x = 8s - 198, & \text{adeoque erit } s > \frac{198}{8} = 24\frac{6}{8}. \end{cases}$$

Conſiſtere igitur debet $s$ inter limites $39\frac{3}{5}$ et $24\frac{6}{8}$, et quoniam $y = \frac{198 - 5s}{3}$ et $x = \frac{8s - 198}{3}$, quo integri eruantur utriuſque valores, debet inſuper $s$ eſſe ex multiplis denominatoris 3. Erunt igitur,

$$\begin{aligned} s &= 27, \ 30, \ 33, \ 36, \ 39, \\ x &= 6, \ 14, \ 22, \ 30, \ 38, \\ y &= 21, \ 16, \ 11, \ 6, \ 1. \end{aligned}$$

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## PROB. IV.

111. *Valentibus nummis aureis quibuſdam* $18\frac{1}{4}$ *ſolidos*, $22\frac{3}{4}$ *ſol. et* $29\frac{1}{4}$ *ſol. reſpective, quot cujuſque claſſis ſumendi ut omnium ſumma ſit* 1000, *valor autem* 1000 l. *Sive, deſignantibus* x, y, *et* z *nummos cujuſque claſſis quæſitos, invenire valores horum integros et affirmativos in æquationibus* $x + y + z = 1000$, *et* $73x + 91y + 117z = 80{,}000$.

Nam multiplicando æquationem $x + y + x = 1000$, ſucceſſive per 73 & 117,

$$\text{erunt} \begin{cases} 73x + 73y + 73z = 73000 \\ 117x + 117y + 117z = 117000 \end{cases}$$

Deinde ſubducendo priorem ab æquatione $73x + 91y + 117z = 80{,}000$, & hanc a poſteriore,

$$\text{erunt} \begin{cases} 18y + 44z = 7000 \\ 44x + 26y = 37000 \end{cases}$$

$$\text{ſive} \begin{cases} 9y + 22z = 3500, \text{adeoq.}\ y = \frac{3500 - 22z}{9}\ \text{et}\ z = \frac{3500 - 9y}{22} \\ 22x + 13y = 18500, \quad - \quad y = \frac{18500 - 22x}{13}. \end{cases}$$

Sed $\frac{3500 - 22z}{9}$ $(= y)$ debet eſſe int. eſt autem $\frac{3501 - 18z}{9}$ int. adeoq. diff. $\frac{4z + 1}{9} = a$ int. adeoq. $z = \frac{9a - 1}{4}$. Similiter, erit differ. $\frac{a - 1}{4} = b$ int. adeoq. $a = 4b + 1$, adeoq. ſubſtit.

$$z = \frac{36b + 8}{4} = 9b + 2. \quad \text{Itemque}$$



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$\frac{18500-22x}{13}$ $(=y)$ debet eſſe int. eſt autem $\frac{18499-26x}{13}$ int. adeoq. diff. $\frac{4x+1}{13}=c$ int. adeoq. $x=\frac{13c-1}{4}$. Et rurſus erit differ. $\frac{c-1}{4}=d$ integer, $\therefore\ c=4d+1$, et ſubſtit.

$$x=\frac{52d+12}{4}=13d+3.$$ Itemque

$\frac{3500-9y}{22}$ $(=z)$ debet eſſe int. eſt autem $\frac{3498}{22}$ int. adeoque

$$\frac{9y-2}{22}=e \text{ int. adeoque } y=\frac{22e+2}{9}, \text{ adeoque}$$

$$\frac{4e+2}{9}=f \text{ int. adeoque } e=\frac{9f-2}{4}, \text{ adeoque}$$

$$\frac{f-2}{4}=g \text{ integer, adeoque } f=4g+2, \text{ et}$$

$$e=\frac{36g+16}{4}=9g+4, \text{ adeoque}$$

$$y=\frac{22\times\overline{9g+4}+2}{9}=22g+10.$$

Inveſtigatis jam valoribus $z$, $x$, et $y$ in terminis $b$, $d$, et $g$ reſpective; ob $9b+2=z$, manifeſtum eſt quod $b$ deſignare poteſt aut zero aut integrum quemlibet affirmativum: quod perinde quoque verum erit de $d$ et $g$. Hiſce autem inter ſe collatis, eruentur limites quidam valorum affirmativorum $b$, $d$, et $g$.

Nam, $22g+10\ (=y)=\frac{3500-22\times\overline{9b+2}}{9}=384-22b$.

Adeoq. $22g+22b=374$.
Sive - $g+b=17$. Et proinde $b$ et $g$ excedere nequeunt 17 reſpective.

Itemque $22g+10=\frac{18500-22\times\overline{13d+3}}{13}=1418-22d$,
Adeoque - $g+d=64$. Unde $d$ nequit eſſe major quam 64 nec minor quam 47.

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Erit igitur $b = 0,\ 1,\ \&c.\ 17$, adeoq. $z = 2,\ 11,\ \&c.\ 155$,

$g = 17, 16, \&c.\ 0,$ - $y = 384, 362, \&c.\ 10,$

$d = 47, 48, \&c.\ 64,$ - $x = 614, 627, \&c.\ 835.$

## P R O B. V.

112. *Invenire valores integros et affirmativos* x, y, *et* z *in æquatione* $17x + 19y + 21z = 400$.

Erit $x = \frac{400 - 19y - 21z}{17}$ int. ſed $\frac{408 - 17y - 17z}{17}$ eſt int.

$\therefore$ differ. $\frac{8 + 2y + 4z}{17}$ integer $= a$,

adeoq. $y = \frac{17a - 4z - 8}{2}$ int. $\therefore \frac{a}{2}$ int. $= b$, $\therefore a = 2b$.

Unde ſubſtituendo,

erit $y = \frac{17 \times 2b - 4z - 8}{2} = 17b - 2z - 4$,

et $x = \frac{400 - 19 \times \overline{17b - 2z - 4} - 21z}{17} = 28 + z - 19b$.

In hiſce valoribus $y$ et $x$, definiendi ſunt $z$ et $b$. Quo determinentur igitur limites $z$, obſervandum eſt, quod $z$ nequit eſſe minor unitate, et ſi pro $x$ et $y$ ſubſtituatur in æquationem datam minimus utriuſque valor ſcilicet 1, evadet $z = \frac{400 - 17 - 19}{21} = 17\frac{7}{21}$, et proinde nequit eſſe major quam 17.

Sit nunc $z = 1$, evadet $y = 17b - 6$, et $x = 29 - 19b$. At $b$ nequit eſſe zero aut negativus, ſic enim prodiret negativus valor. $y$; neque tranſcendere poteſt $\frac{29}{19} = 1\frac{10}{19}$, ſic

enim foret $x$ negativus. Unicus igitur valor $b$ erit 1, et in hoc caſu $y = 17 - 6 = 11$, et $x = 29 - 19 = 10$. Similiter, ſit $z = 2, 3, 4, 5, 6$ ſucceſſive, erit ut ante 1, unicus valor qui dabit $y$ et $x$ affirmativos, et in his caſibus erit $y = 9, 7, 5, 3, 1$, et $x = 11, 12, 13, 14, 15$, reſpective.

Sit $z = 7, 8, 9, 10$ ſucceſſive, nullum ſortietur valorem $b$, qui non dabit $y$ et $x$ negativum, et inde rejiciendi ſunt hi valores $z$.

Sit $z$ autem $= 11, 12, 13, 14$, poſito $b = 2$, evadet $y = 8, 6, 4, 2$, et $x = 1, 2, 3, 4$ reſpective. Reliqui valores $z$ ſcilicet 15, 16, 17, perinde rejiciendi ſunt. Reſtant igitur decem valores integri et affirmativi $x$, $y$, et $z$, ſcil.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $z =$ | 1, | 2, | 3, | 4, | 5, | 6, | 11, | 12, | 13, | 14, |
| $y =$ | 11, | 9, | 7, | 5, | 3, | 1, | 8, | 6, | 4, | 2, |
| $x =$ | 10, | 11, | 12, | 13, | 14, | 15, | 1, | 2, | 3, | 4. |

## P R O B. VI.

113. *Invenire valores integros et affirmativos* x, y, *et* z *in æquatione* $5x + 7y + 11z = 224$.

Erit $x = \frac{224 - 7y - 11z}{5}$ int. ſed, $\frac{225 - 5y - 10z}{5}$ eſt int.

$$\therefore \frac{1 + 2y + z}{5} = a \text{ integer,}$$

adeoq. $y = \frac{5a - z - 1}{2}$ $\therefore$ $\frac{a - z - 1}{2} = b$ integer,

et $a = 2b + z + 1$. Unde ſubſtit.

erit $y = \frac{5 \times \overline{2b + z + 1} - z - 1}{2} = 5b + 2z + 2$,

Et $x = \frac{224 - 7 \times \overline{5b + 2z + 2} - 11z}{5} = 42 - 7b - 5z$.

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Cæterum $z$ nequit eſſe minor unitate, nec major quam $\frac{224-5-7}{11} = 18\frac{4}{11}$, adeoque limites 1 et 18 haud tranſgredietur, ſit

| | | | | |
|---|---|---|---|---|
| $z=1$, erit | $y=5b+4$, et | $x=37-7b$, | adeoq. | $b=0,1,2,3,4,5,$ |
| $z=2$, | $y=5b+6$, | $x=32-7b$, | - | $b=-1,0,1,2,3,4,$ |
| $z=3$, | $y=5b+8$, | $x=27-7b$, | - | $b=-1,0,1,2,3,$ |
| $z=4$, | $y=5b+10$, | $x=22-7b$, | - | $b=-1,0,1,2,3,$ |
| $z=5$, | $y=5b+12$, | $x=17-7b$, | - | $b=-2,-1,0,1,2,$ |
| $z=6$, | $y=5b+14$, | $x=12-7b$, | - | $b=-2,-1,0,1,$ |
| $z=7$, | $y=5b+16$, | $x=7-7b$, | - | $b=-3,-2,-1,0,$ |
| $z=8$, | $y=5b+18$, | $x=2-7b$, | - | $b=-3,-2,-1,0,$ |
| $z=9$, | $y=5b+20$, | $x=-3-7b$, | - | $b=-3,-2,-1,$ |
| $z=10$, | $y=5b+22$, | $x=-8-7b$, | - | $b=-4,-3,-2,$ |
| $z=11$, | $y=5b+24$, | $x=-13-7b$, | - | $b=-4,-3,-2,$ |
| $z=12$, | $y=5b+26$, | $x=-18-7b$, | - | $b=-5,-4,-3,$ |
| $z=13$, | $y=5b+28$, | $x=-23-7b$, | - | $b=-5,-4,$ |
| $z=14$, | $y=5b+30$, | $x=-28-7b$, | - | $b=-5,$ |
| $z=15$, | $y=5b+32$, | $x=-33-7b$, | - | $b=-6,-5,$ |
| $z=16$, | $y=5b+34$, | $x=-38-7b$, | - | $b=-6,$ |
| $z=17$, | $y=5b+36$, | $x=-43-7b$, | - | $b=-7,$ |
| $z=18$, | $y=5b+38$, | $x=-48-7b$, | - | $b=-7.$ |

Dantur itaque 59 valores $b$, quibus ſubſtitutis totidem prodeunt valores integri et affirmativi $y$ et $x$, ſcil.

| | | |
|---|---|---|
| $z=1$, | $y=4, 9, 14, 19, 24, 29,$ | $x=37, 30, 23, 16, 9, 2,$ |
| $z=2$, | $y=1, 6, 11, 16, 21, 26,$ | $x=39, 32, 25, 18, 11, 4,$ |
| $z=3$, | $y=3, 8, 13, 18, 23,$ | $x=34, 27, 20, 13, 6,$ |
| &c. | &c. | &c. |
| $z=18$, | $y=3$, | $x=1.$ |

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## PROB. VII.

114. *Quot modis erogare licet* 1000l. *adhibitis ſolummodo nummis* 5 *ſolidorum*, 21 *ſol. et* 27 *ſol. reſpectivè. Seu quæruntur omnes valores poſſibiles* x, y, et z *in æquatione* $5x + 21y + 27z = 20{,}000$.

Erit $x = \frac{20{,}000 - 21y - 27z}{5} \therefore \frac{y + 2z}{5} = a$ int. et $y = 5a - 2z$,

et $x = \frac{20{,}000 - 21 \times \overline{5a - 2z} - 27z}{5} = 4000 - 21a + 3z$.

Cæterum non poſtulat quæſtio $x$, $y$, et $z$ in omni caſu ſimul eſſe adhibendos, poteſt igitur horum quivis evadere zero, unde erit minimus valor $z = 0$, et maximus valor, evaneſcentibus ſcilicet $x$ et $y$, $\frac{20{,}000}{27} = 740$. His cognitis, innoteſcunt limites $a$, nam

$$5a - 2z \begin{Bmatrix} \text{ſi } z = 0 \\ \text{ſi } z = 740 \end{Bmatrix} \begin{matrix} = 5a, \text{ unde prodit minor limes } a = 1 \\ = 5a - 1480, \text{ et major limes } a = \end{matrix}$$

$$= \frac{1480}{5} = 296.$$

| | | | valores z. | |
|---|---|---|---|---|
| Sit $a = 1$, erit | $y = 5 - 2z$, et | $x = 3979 + 3z$, | 3 | ſcil. 0, 1, 2, |
| $a = 2$, | $y = 10 - 2z$, | $x = 3958 + 3z$, | 6 | 0, 1, 2, 3, 4, 5, |
| $a = 3$, | $y = 15 - 2z$, | $x = 3937 + 3z$, | 8 | 0, 1, &c. |
| $a = 4$, | $y = 20 - 2z$, | $x = 3916 + 3z$, | 11 | 0, 1, &c. |
| $a = 5$, | $y = 25 - 2z$, | $x = 3895 + 3z$, | 13 | 0, 1, &c. |
| &c. | &c. | &c. | | |

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Manifeſtum eſt quod valores *z* reſtringuntur tantum per ſeriem valorum *y* donec numerus abſolutus in ſerie valorum *x* decreſcendo evaſerit negativus, quod fiet poſt terminum progreſſionis $\frac{3979}{21} + 1 = 190$. Sit

| | | | valores *z*. |
|---|---|---|---|
| $a = 189$ | $y = 945 - 2z$ | $x = 31 + 3z$ | $473$ |
| $a = 190$ | $y = 950 - 2z$ | $x = 10 + 3z$ | $476$ |
| $a = 191$ | $y = 955 - 2z$ | $x = -11 + 3z$ | $477 - 3 = 474$ |
| $a = 192$ | $y = 960 - 2z$ | $x = -32 + 3z$ | $480 - 10 = 470$ |
| $a = 193$ | $y = 965 - 2z$ | $x = -53 + 3z$ | $482 - 17 = 465$ |
| $a = 194$ | $y = 970 - 2z$ | $x = -74 + 3z$ | $485 - 24 = 461$ |
| $a = 195$ | $y = 975 - 2z$ | $x = -95 + 3z$ | $487 - 31 = 456$ |
| &c. | &c. | &c. | &c. |
| $a = 295$ | $y = 1475 - 2z$ | $x = -2195 + 3z$ | $737 - 731 = 6$ |
| $a = 296$ | $y = 1480 - 2z$ | $x = -2216 + 3z$ | $740 - 738 = 2.$ |

Ad terminum 190 *um* uſque liquet quod inter valores *z*, cenſeri poteſt zero, ſalvis valoribus poſſibilibus *y* et *x*, poſtea vero non item, ex eo quod valores *x*, ob terminum abſolutum jam negativum, emergent negativi ſeu impoſſibiles. Porro, ex valoribus ſubſequentibus *z* ope progreſſionis *y* elicitis tot auferri debent quot pariunt negativos valores *x*. Sic in termino 191 *mo*, tres valores *z*, ſcilicet 1, 2, 3 qui negativum exhibent $-11 + 3z = x$ demendi ſunt a 477 valoribus ope $955 - 2z = y$ comparatis, et inde reſtant 474 tantum valores.

Denique, notandum eſt quod termini 190 valorum *z*, ex binis progreſſionibus arithmeticis conflantur, ſcilicet 3, 8,



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13, &c. et 6, 11, 16, &c. Ex binis etiam termini reliqui 296 — 190 = 106, ſcilicet 474, 465, 456, &c. et 470, 461, 452, &c. Quatuor autem progreſſionum ſumma dabit omnes valores $z$ et idcirco problematis ſolutiones,

$$3,\ 8,\ 13,\ -\ 473 = \frac{\overline{3+473} \times 95}{2} = 22610$$

$$6,\ 11,\ 16,\ -\ 476 = \frac{\overline{6+476} \times 95}{2} = 22895$$

$$474,\ 465,\ -\ 6 = \frac{\overline{474+6} \times 53}{2} = 12720$$

$$470,\ 461,\ -\ 2 = \frac{\overline{470+2} \times 53}{2} = 12508$$

$$70733$$

115. Huic quæſtioni 70734 ſolutiones tribuit *Simpſon Algeb. p.* 200, edit. 2dæ. Virum acutiſſimum ſane et cui multum debet Matheſis, præ feſtinatione effugit, zero minorem limitem $q$ (ſeu $a$) deſignare non poſſe. Hoc enim poſito, valor generalis $y$, ſcilicet $5q - 2z$ evaderet negativus ſeu impoſſibilis. Pariter lapſus eſt ſtatuendo ſolutiones 60 problematis penultimi, 59 tantum admittentis; nam ubi $z = 14$, unicum ſortietur valorem $y$, ſcilicet 5, et $x$, ſcilicet 7. Sed

*Opere in longo fas eſt obrepere ſomnum.*

116. Si quærantur valores integri et affirmativi, $x$, $y$ et $z$, quæſtio 70395 ſolutiones ſortietur, nam quatuor progreſſiens arithmeticæ evadent

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| | | | |
|---|---|---|---|
| 2, 7, 12 | - - - | 472 | = 22515 |
| 4, 9, 14 | - - - | 474 | = 22705 |
| 474, 465, 456 | - | 6 | = 12720 |
| 469, 460, 451 | - | 1 | = 12455 |
| | | | 70395. |

117. Notandum eſt quod in quæſtionibus trium quantitatum incognitarum ſolvendis a methodo *Moivreana* defleximus, utpote quæ ſolutiones juſto pauciores elicitura ſit, niſi quantitas *z* in generalibus valoribus *y* et *x*, prodeat iiſdem affecta ſignis; aut ſubſtituendo valorem binomium ad idem in utriſque revocetur ſignum: cæterum haud raro prodibit *z* diverſis affecta ſignis, ut in exemplis allatis; et aliquando nullo artificio, notante *Dodſon*, *p.* 22, ad eadem ſigna in utriſque revocari poteſt.

His veſtigiis inſiſtendo eruere poſſimus ſolutiones æquationum quatuor quantitates incognitas involventium, cæterum quum parum utiles ſint, adjungam potius problema cujus uſus in Chronologia egregius.

## P R O B. VIII.

118. *Invenire numerum quendam qui per diviſores duos pluresve datos ſucceſſive diviſus, reſidua ſortietur numeris quoque datis reſpective æqualia.*

*Ex. gr.* Quotus annus *Periodi Juliani* erit ſimul tertius *Cycli Solis*, ſeptendecimus *Cycli Lunæ*, et undecimus *Cycli Indictionis*; deſignantibus nimirum Cyclis hiſce annorum 28, 19, et 15 reſpective, diviſores datos, et numeris 3, 17, et 11, reſidua data.



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Sit $n$ numerus qui per 28 et 19 diviſus ſortietur reſidua 3 et 17 reſpective, eruntque $\frac{n-3}{28}$ et $\frac{n-17}{19}$ integri. Finge $x = \frac{n-3}{28}$ et $y = \frac{n-17}{19}$. Unde prodibit $n = 19y + 17 = 28x + 3$, et $y = \frac{28x - 14}{19}$ integer, dempto autem integro $\frac{19x - 19}{19}$ erit reſiduum $\frac{9x + 5}{19} = a$ integer, adeoque $x = \frac{19a - 5}{9}$ integer, et reſiduum $\frac{a + 4}{9} = b$ integer, et $a = 9b - 4$. Unde ſubſtituendo, erit $x = 19b - 9$, $y = 28b - 14$, et $n = 532b - 249$. Hinc erit $b$ affirmativus, & ſubſtituto minimo valore ejus, ſcilicet 1, evadet $n = 532 - 249 = 283$, qui utique conditiones diviſorum 28 et 19 implebit.

Porro, ſi huic numero 283 addatur aut diviſor primus 28, aut multiplus ejus quivis, adhuc explebit ſumma conditionem diviſoris primi. Similiter, ſi eidem 283 addatur aut diviſor ſecundus 19, aut multiplus ſecundi quilibet, explebit ſumma conditionem diviſoris ſecundi; et proinde ſi eidem addatur 532 factum ex utriſque diviſoribus in ſe ductis, aut quivis multiplus ejus, puta $532z$, deſignante nimirum $z$ integrum quemcunque affirmativum, perinde explebit ſumma diviſoris utriuſque conditiones.

Finge nunc $532z + 283$ conditionibus diviſoris primi et ſecundi ſatisfacientem, eſſe numerum quæſitum, atque idcirco conditioni tertii quoque diviſoris 15 reſpondere; eritque $\frac{532z + 283 - 11}{15}$ integer, adeoque dempto integro

$\frac{525z+270}{15}$ erit $\frac{7z+2}{15}$ integer $= c$, et $z = \frac{15c-2}{7}$; atque iterum $\frac{c-2}{7} = d$ integer, et $c = 7d + 2$: unde ſubſtituendo erit $z = 15d + 4$. Hinc erit $d$ vel zero, vel affirmativus; et minimus valor $z$, evaneſcente $d$, $= 4$. Evadet igitur numerus quæſitus $532z + 283 = 2128 + 283 = 2411$, qui nimirum annum periodi Juliani quæſitum indicabit.

Similiter, Datis diviſore quarto reſiduoque ſibi proprio, invenire licet numerum qui conditionibus quatuor diviſorum ſatisfaciet, ſi numero jam invento 2411 addatur multiplus quidam 7980 ex tribus diviſoribus datis conflati; atque ita deinceps.

Sin numerus hoc pacto inventus minimum multiplum communem omnium diviſorum datorum tranſcendat, non erit minimus qui cunctas explebit conditiones; dividendo autem hunc numerum per minimum multiplum communem, reſiduum peracta diviſione dabit minimum. Et ſi huic reſiduo addatur deinceps diviſor communis, inveniri poſſunt ad libitum numeri qui problematis conditiones exſolvent.

119. Traditis jam ſolutionibus problematum indeterminatorum, quæ ad æquationes ſimplices deducunt, nonnulla ex *Diophantæis*, quæ circa numeros quadratos verſantur, et ex quadraticarum reductione pendent, adjungere libet.

De numeris rationalibus tantum agit *Diophantus* ablegatis ſurdis.

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## PROB. I.

120. *Numerum quadratum datum in duos quadratos dividere.*

Sit quadratus dividendus 25, et q. quæſiti $x^2$ et $y^2$, adeoque $x^2 + y^2 = 25$. Finge jam latus alterutrius quadrati quæſiti eſſe æquale ſummæ vel differentiæ lateris quadrati dati et lateris alterius quæſiti quolibet numero præter unitatem affecti, puta $y = 5 \pm 2x$ vel $2x \pm 5$. Quadrando igitur hunc valorem $y$, et ſubſtituendo pro $y^2$, evadet

$$5x^2 \pm 20x + 25 = 25.$$

Et tranſponendo $5x^2 = \mp 20x$

Et dividendo - - $x = \mp 4$, adeoque $y (= 5 \pm 2x) = \mp 3$.

Et horum quadrata 16 et 9 conficiunt 25.

Similiter, ſit $y = 5 - \frac{1}{2}x$, per reductionem evadet $x = 4$ et $5 - \frac{1}{2}x = 3$. Sit $y = 5 - 3x$, erit $x = 3$ et $5 - 3x = -4$. Sit $y = 4x - 5$, eritque $x = \frac{40}{17}$ et $4x - 5 = \frac{75}{17}$ quorum quadrata $\frac{1600}{289}$ et $\frac{5625}{289}$ abſolvunt 25. Atque ita deinceps. Sed quo magis ab unitate recedunt coefficientes $x$, in valores fractos eo profuſiores excurrent latera quæſita.

121. Quo percipiatur ſingulare *Diophanti* artificium in hac ſolutione, obſervandum eſt, $1^{mo}$. Latus quadrati dati in membrum unum lateris ficti adſumi, ad tollendum ex æquatione reſultante quadratum datum, quo pacto æquatio

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quadratica, ſublato termino abſoluto deprimetur ad ſimplicem, et inde eruetur valor $x$ per diviſionem ſolam, qui idcirco ſemper erit rationalis.

2. Debere coefficientem $x$ membri alterius lateris ficti eſſe majorem minoremve unitate; nam ſi ponatur unitas, orietur valor $x$ in æquatione reſultante æqualis lateri ipſius quadrati dividendi, et proinde ipſum latus fictum reperietur nihil. Sic poſito $y = 5 - x$, per reductionem prodibit $x = 5$, et $5 - x = 0$.

3. Poſſe latus fictum eſſe binomium vel reſiduum, ex priore enim emergent latera negativa, ex poſteriore eadem poſitiva, quorum utraque æquo jure quæſtionem ſolvent; utplurimum vero ponitur reſiduum, quam formam ipſe adhibet *Diophantus*.

## PROB. II.

122. *Numerum datum ex duobus quadratis conflatum in alios duos quadratos dividere.*

Sit numerus dividendus $13 = 4 + 9$. Sumptis lateribus quadratorum datorum 2 et 3, ponantur latera quæſitorum quadratorum $x - 2$ et $2x - 3$. Horum quadrata ſimul $5x^2 - 16x + 13 = 13$, et per reductionem prodibit $x = \frac{16}{5}$; adeoq. $x - 2 = \frac{6}{5}$ et $2x - 3 = \frac{17}{5}$, quorum quadrata $\frac{36}{25}$ et $\frac{289}{25}$ ſimul $= \frac{325}{25} = 13$. At ſi ponantur latera $x + 2$ et $2x - 3$, erit $5x^2 - 8x + 13 = 13$, et per reductionem $x = \frac{8}{5}$ adeoque $x + 2 = \frac{18}{5}$ et $2x - 3 = \frac{1}{5}$

at horum quadrata quoque $\frac{324}{25}$ et $\frac{1}{25}$ conficiunt 13. Atque ita ſemper prodibunt bina quadratorum paria, quorum alterutrum quæſtioni ſatisfaciet, prout latera ficta ponantur vel utraque reſidua, vel alterum binomium exiſtente altero reſiduo.

123. Cæterum cavendum eſt 1^mo^. Ne idem ſtatuatur coefficiens $x$ in utroque latere ficto; alioquin latera quadratorum quæſitorum non prodibunt diverſa a datis. Sic poſitis $x \mp 2$ et $x - 3$, evadent latera quæſita 3 et $\pm 2$ quæ jam data ſunt.

2. Ne coefficientes $x$ in utroque latere, ubi hæc latera ſunt reſidua, ſint inter ſe in ratione laterum datorum, aut altero exiſtente binomio, in ratione ſummæ datorum laterum ad eorundem differentiam; alioquin idem ſequetur abſurdum, quod ſcilicet latera quæſita emergent eadem ac data.

## P R O B. III.

124. *Invenire numerum quadratum qui adjecto quolibet numero conficiat quadratum.*

Sit quadratus quæſitus $x^2$, conficiatque quadratum $x^2+3$. Finge latus poſterioris, $x - n$ (deſignante $n$ numerum quemcunque) et quadrando, erit

$$x^2 - 2nx + n^2 = x^2 + 3,$$

Et tranſponendo - $2nx = n^2 - 3$

Adeoque - - - - - - $x = \frac{n^2 - 3}{2n}$.

Poſito igitur $n = 6$, erit $x = \frac{11}{4}$, adeoq. $x^2 = \frac{121}{16}$, et $x^2 + 3 = \frac{169}{16}$ quadratus nimirum cujus latus eſt $\frac{13}{4}$. Variante autem utcunque $n$, variari quæſtionis ſolutiones manifeſtum eſt. Similiter, *invenire licet quadratum a quo dempto numero dato reſiduum erit quadratus.*

125. In quæſtionibus prioribus negotium erat æquationem quadraticam ad ſimplicem deprimere, tollendo quadratum datum, in noviſſimâ vero, quadratum quæſitum. Æquationum quadraticarum reductionem nuſquam in ſex libris *Arithmeticorum* jam extantibus ex profeſſo tradit *Diophantus*, harum tamen regulam tali latuiſſe viro minime credendum eſt; & ipſe quidem pollicetur, Def. 11. ſe quadraticarum ſolutiones poſtea commonſtraturum. Satis profectò evincunt problematis inſequentis limitationes, cujus ſolutionem mira ſubtilitate expedit *Diophantus*, Lib. 5, Quæſt. 33, illum probe quadraticarum reductionem cognoviſſe.

Ὀκταδράχμους καὶ πενταδράχμους χοέας τὶς ἔμιξε
  Τοῖς προπόλοισι πιεῖν χρῆστ᾽ ἀποταξάμενος·
Καὶ τιμὴν ἀπέδωκεν ὕπερ πάντων τετράγωνον,
  Τὰς ἐπιταχθείσας δεξάμενον μονάδας,
Καὶ ποιοῦντα πάλιν ἕτερον σε φέρειν τετράγωνον,
  Κτησάμενον πλευρὰν σύνθεμα των χοέων.
Ὥστε διάστειλον τοὺς ὀκταδράχμους πόσοι ἦσαν
  Καὶ πάλι τοὺς ἑτέρους παῖ, λέγε πενταδράχμους.



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Epigrammatis propoſitum tale eſt. Emit quidam duos cados vini, unius quidem choam drachmis 8, alterius vero choam drachmis 5, in uſum famulorum miſcendos; et pro omnibus chois perſolvit numerum quadratum, cui addendo numerum datum 60, fit alter quadratus, latus habens ſummam choarum. Quæritur numerus choarum utriuſque pretii.

Sit $x$ ſumma choarum, et cum pretium ſummæ + 60 adæquat $x^2$, erit pretium ſummæ $x^2 - 60$. Finge hoc erogari pro chois 5 drachmarum ſolis, et prodibit numerus choarum quinque drachmis $\frac{x^2 - 60}{5}$ major quam $x$, (cum $x$ contineat choas aliquot 8 drachmis) erit igitur $x^2 - 60$ major quam $5x$, ſive $x^2$ major quam $5x + 60$, et per reductionem, $x$ major quam $\frac{5}{2} + \sqrt{\frac{25}{4} + 60}$, id eſt, major quam 10,7. Similiter, erit numerus choarum ſolis 8 drachmis $\frac{x^2 - 60}{8}$ minor quam $x$, adeoque $x^2$ minor quam $8x + 60$, et per reductionem, $x$ minor quam 12,7. Oportet igitur valorem $x$ inter limites 10,7, et 12,7 inveniri, et proinde *a fortiori*, inter limites integros arctiores 11 et 12, quos ponit *Diophantus*.

Sed altera quoque datur reſtrictio valoris $x$; ſcilicet, quod $x^2 - 60$ debet eſſe numerus quadratus. Finge igitur latus hujus $x - y$, et quadrando, erit $x^2 - 2xy + y^2 = x^2 - 60$, adeoque per prob. *præced.* $x = \frac{y^2 + 60}{2y}$. At neceſſe eſt $\frac{y^2 + 60}{2y}$ $(= x)$ eſſe majorem quam 11, ſive $y^2$ majorem quam $22y - 60$, adeoque per reductionem, $y$ majo-

rem quam 11 + $\sqrt{61}$ ſive 18,7. et pari ratione, $\frac{y^2+60}{2y}$ eſſe minorem quam 12, ſive $y^2$ minorem quam $2y - 60$, adeoque per reductionem, $y$ minorem quam 21,2. Oportet igitur numerum aſſumptum $y$ inter limites 18,7 et 21,2. conſiſtere, adeoque inter limites integros *Diophantæos* 19 et 21.

Puta jam $y$ medium obtinere locum inter 19 et 21, ſcilicet 20, et ſubſtituto hoc valore $y$ in æquationem $x^2 - 2xy + y^2 = x^2 - 60$, prodibit $x^2 - 40x + 400 = x^2 - 60$, adeoque $x = 11\frac{1}{2}$, et proinde $x^2 - 60 = 132\frac{1}{4} - 60 = 72\frac{1}{4}$. Ergo ſumma choarum fit $11\frac{1}{2}$, intra limites nempe præſcriptos, et totum pretium $72\frac{1}{4}$ drachmarum, ſcilicet numerus quadratus (utpote latus habens $8\frac{1}{2}$.)

Denique, quo inveniatur numerus choarum utriuſque pretii, ſit $z$ numerus choarum 5 drachmis, et erit $11\frac{1}{2} - z$ numerus choarum 8 drachmis, pretium priorum erit $5z$ drachm. poſteriorum autem $92 - 8z$ drachm. adeoque pretium ſummæ $5z + 92 - 8z$ ſive $92 - 3z = 72\frac{1}{4}$, et proinde $3z = 19\frac{3}{4}$, et $z = 6\frac{7}{12}$, et inde $11\frac{1}{2} - z = 4\frac{11}{12}$. Numerus igitur choarum 5 drachmis evadit $7\frac{7}{12}$, choarum autem 8 drachmis $4\frac{11}{12}$.

Alii quoque dantur limites valoris $y$. Nam $11 \pm \sqrt{61}$ evadit 18,7 vel 3,3. et $12 \pm \sqrt{84}$ evadit 21,2 vel 2,8. Conſiſtere igitur poteſt $y$ inter 3,3 et 2,8. Puta $y = 3$, et ſubſtituendo $x - 3$ ut antea, exurget $x = 11\frac{1}{2}$. Sin limites alterutros tranſgrediatur $y$, exulabit $x$ a limitibus præſcriptis, et numerus choarum vel 8 drachmis vel 5 drachmis emerget negativus, quod impoſſibile eſt.



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126. Pauca hæc exempla in ſpecimen ſagaciſſimi *Diophanti* ingenii ad quæſtiones vel ſpinoſiſſimas expediendas ſufficiant. Haud parum quidem intentionis animi lectoribus etiam mathematice doctis requiritur ad ſolutiones complures *Diophantæas* capiendas, præter materiæ ſubtilitatem, ex eo quod perpaucas notas algebraicas adhibet, adeo ut plures ſæpe quantitates incognitæ eadem deſignantur ſpecie. Sic in propoſitione noviſſima, quantitates quas notis $x$, $y$, et $z$, diſtinximus, omnes una ſpecie $s$ deſignat *Diophantus*.

127. Quo tempore inventa fuit Arithmetica ſpecioſa ſeu Algebra, in incerto eſt. Hoc ſaltem præſumendum eſt, inventorem non fuiſſe *Diophantum*, quippe qui medias in res, haud ſecus ac notas lectorem rapit, et nullis fere præmiſſis principiis, quæſtionibus ſolvendis ſtatim ſe accingit. Innuit tantum, idque obiter, quaſi demonſtratione non egens, "quod minus in minus dat plus, in plus autem, minus." Et præterea monet, Def. 10, "Expedit eum qui hoc negotii ſit aggreſſurus, in additione ſubductione et multiplicatione ſpecierum jamdudum exercitatum eſſe." Niſi forſan inſtituebantur elementa artis neceſſaria in alio quodam opere *Diophanti* jam deperdito, de *Arithmeticæ praxi*; de quo meminit *Theon*.

128. Priſcis Mathematicis *Platoni*, *Euclidi*, *Archimedi*, &c. hanc Artem innotuiſſe, recentiorum contenderunt multi. *Schootenius*, *Nonius* aliique, ad quorum ſententiam accedit quoque *Wallis*, cum certo ſcirent ex demonſtrationum et conſtructionum ſubtilitate et elegantia, analyſin *quandam* Veteribus probe cognitam eſſe, neque aliam noſſent præter eam quæ Algebræ ſpecioſæ ope inſtituitur, nullumque hujus veſtigium ante *Diophantum* inveniſſent; Veteres admirationis aucupandæ temere inſimularunt, utpote qui inventa ſua vulgari tantum ſyntheſeos forma tradebant,

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hoc conſilio, ut artem ſuam analyticam a Poſteris celarent. Luculenter autem conſtat ex *Pappi Alexandrini* præfatione ad Lib. 7, *Collectionum Mathematicarum*, veteres Geometras in ſolutionibus Theorematum & Problematum propoſitorum inveſtigandis analyſin *geometricam* uſurpaſſe; eamque plenius expoſuiſſe *Euclidem*, nempe Elementorum ſcriptorem, *Apollonium Pergæum; Ariſtæum* ſeniorem, et *Eratoſthenem;* quorum opera analytica 33 libris contenta recenſet *Pappus*, qui & ipſe naturam analyſeos & ſyntheſeos Veterum optime explicat.

Analyſin Veterum ita collaudat *Halley* Geometer ſimul et Algebraiſta ſummus, in Præfatione ad editionem ſuam *Apollonii Pergæi*, *De Sectione Rationis*.

" Methodus hæc cum *Algebra Speciosa* facilitate conten-
" dit, evidentiâ vero & demonſtrationum elegantiâ eam
" longe ſuperare videtur : ut abunde conſtabit, ſi quis con-
" ferat hanc *Apollonii* doctrinam *De Sectione Rationis* cum
" ejuſdem Problematis analyſi Algebraicâ, quam exhibuit
" *Cl. Walliſius*, *Tom* 2, *Operum Mathemat. cap.* 54, *pag.* 220."

Conſulat quoque lector Appendicem ad *Simpſoni Conica*, *edit.* 2*da*. ubi trium propoſitionum ſolutiones Algebraicas cum iis quæ per analyſin geometricam fiunt conferre licet. Miro certe veteris analyſeos flagrabat amore *Halley*, qui immane quoddam, imo vix credibile eſt auſus, nempe opus interpretandi Verſionem Arabicam *Apollonii de Sectione Rationis* (*Apollonii* opere περὶ λόγυ ἀποτομῆς deperdito) linguæ Arabicæ ipſe prorſus ignarus ! & quod vel Arabice doctiſſimum abſterruerit, codex ille peſſimâ manu exaratus, punctiſque diacriticis deſtitutus, & diagrammatum lineas literis male ſignatas & diſtinctas habens, mendis lacuniſque ſcatebat. Verùm beneficio ſchedarum aliquot, quas, paucis ante annis, latine verterat *D. Bernardus*, Aſtronomiæ Profeſſor *Savilianus*, quæ, clavis ad inſtar, aditum illi aperuerunt ad *Apollonii* mentem inveſtigandam ; excerptis vocibus illis de

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quibus ex verſione *Bernardi* liquido conſtabat, dein ad argumentum reſpiciendo, & notas obſcuriores iterum iterumque ſecum evolvendo, quid voluerunt paulatim deprehendit, & hac quaſi *deciphrandi* methodo totum librum perlegit, eundemque denuo pedetentim percurrendo, opus integrum abſque alterius cujuſpiam auxilio ad finem perduxit.

129. Obſervandum reſtat, quod reſolutio æquationum indeterminatarum in applicatione Algebræ ad Geometriam apprime utilis eſt, cum præcipuarum curvarum æquationes ſunt indeterminatæ. Methodus tamen generalis tales ſolvendi hactenus deſideratur ultra problemata primi gradus. Licet enim occurrunt ſolutiones problematum ſecundi, tertii et etiam quarti gradus apud *Diophantum* ejuſque Commentatores, ſunt tantum particulares. Qui plura autem de hiſce deſiderat, conſulat diſſertationes *Euleri*, *Acad. Petrop. Tom* 6, *veter.* et *Tom* 9, *nov.* Et *La Grange Acad. Berlin. Tom* 23.

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# ANALYSIS ÆQUATIONUM.

## PARS TERTIA.

130. Hactenus expoſuimus modos ordinandi & concinnandi æquationes, eaſdemque reducendi ſi modo ſecundum non tranſcendant gradum. Antequam vero reſolutionem æquationum dimenſionum ſuperiorum aggrediamur, neceſſe erit *præparationem æquationum* inſtituere, præmittendo aliqua de naturâ radicum & præcipuis tranſmutationibus æquationum.

### De natura radicum æquationis.

131. *Radix eſt numerus vel ſpecies qui ſi in æquatione pro ſpecie radicem ſignificante ſubſtituatur, efficiet omnes terminos evaneſcere.*

Sic æquationis $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, unitas eſt radix, quoniam ſcripta pro $x$ producit $1 - 1 - 19 + 49 - 30$, hoc eſt nihil. Sed

132. *Æquationis ejuſdem plures eſſe poſſunt radices.*

Nam ſi in hac eadem æquatione $x^4 - x^3$, &c. $= 0$, pro $x$ ſcribas numerum 2, & pro poteſtatibus $x$ ſimiles poteſtates numeri 2, producetur $16 - 8 - 76 + 98 - 30$, hoc eſt nihil. Atque ita ſi pro $x$ ſcribas numerum 3 vel numerum negativum $-5$, utroque caſu producetur nihil, terminis affirmativis & negativis in his quoque caſibus ſe mutuo deſtruentibus. Proinde cum numerorum 1, 2, 3

vel — 5, quilibet ſcriptus in æquatione pro $x$ impleat conditionem ipſius $x$, efficiendo ut termini omnes æquationis conjunctim æquentur nihilo, erit quilibet eorum radix æquationis.

Atque ita æquationis $x^3 = 1$, ſeu $x^3 - 1 = 0$, tres radices erunt 1, $\frac{-1+\sqrt{-3}}{2}$, vel $\frac{-1-\sqrt{-3}}{2}$, nam ſi ſcribas 1 pro $x$, producetur $1 - 1$, hoc eſt nihil; ſi ſcribas $\frac{-1+\sqrt{-3}}{2}$ pro $x$, producetur $\frac{8}{8} - 1$, hoc eſt nihil; et ſimiliter, ſi ſcribatur $\frac{-1-\sqrt{-3}}{2}$ pro $x$. In ſpeciebus quoque idem fiet; ſic æquationis $x^4 - px^3 (-p^2 + q)x^2 + p^3x - p^2q = 0$, ſunt $p$, $-p$, $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$, vel $+ \frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$ radices; nam ſcribendo $p$ pro $x$ producetur $p^4 - p^4 - p^4 + p^2q + p^4 - p^2q$, hoc eſt nihil; & ſic deinceps.

Hinc liquet quantitates reales & impoſſibiles æque fungi poſſe munere radicum, efficiendo omnes æquationis terminos evaneſcere.

133. *Eandem vero æquationem plures habere poſſe radices æquum eſt, cum plures eſſe poſſunt ejuſdem problematis ſolutiones.* § 34.

134. Raro tamen *omnes* æquationis radices, etiamſi *reales & affirmativæ* ſint, problema a quo deducta eſt iſta æquatio ſolvent. § 35 & 44. Sed hoc fit ob limitationem aliquam in problemate quod ad æquationem nil ſpectat. Nam ſi æquatio non complectitur omnes conditiones problematis,

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non neceſſe eſt ut omnium conditionum teneatur limitibus, quicquid amplius eſt in problemate quam in æquatione poteſt illud limitibus arctare, hanc non item. § 32. Veruntamen neceſſe eſt æquationem qua reſpondetur tot habere radices quot ſunt quæſitæ quantitatis caſus diverſi ab iiſdem datis pendentes, & eadem argumentandi ratione determinandi. Nam quiſque caſus diverſus ſolutionem diverſam poſtulat. Et quidem caſus problematum indigitare a datis diverſos, eſt proprium munus radicum *negativarum*. § 43 & 93. Hinc in quæſtionibus geometricis et phyſicis radices affirmativæ & negativæ ad plagas oppoſitas tendere ſolent. § 40, & 42.

135. Hæc æquationum copia ſimul quæſita & non quæſita complectentium inveniendis diverſis caſibus ejuſdem problematis generalis inſervire poteſt. Nihilominus fatendum eſt, incommodo ſæpe laborare hanc copiam, ubi ſolutionem quæſitam aliis nexam atque implicitam non ſine calculo valde operoſo ſecernere atque evolvere licet; ut bene quidem monet *D'Alembert* *. " Les quantités négatives — montrent tout à la fois & la richeſſe de cette ſcience, qui fait trouver dans la ſolution du problême, juſqu' aux choſes qu'on ne demandoit pas, & en même tems, ſi on oſe le dire, l'imperfection du calcul, qui en donnant ce qu'on ne cherche pas, & qu'on ne lui demande point, ne donne pas toujours ce qu'on lui demande avec toute la perfection qu'on pourroit exiger. C'eſt qui n'arrive que trop dans les queſtions algébriques; la ſolution d'un problême, qui n'en a quelquefois réellement qu'une ſeule poſſible (dans le ſens où il a été propoſé) eſt ſouvent incorporée & comme amalgamée

* *Melanges de Literature*, vol. 5, p. 230.

" avec pluſieurs autres ſolutions de problêmes analogues,
" mais differens ; ſolutions qui enveloppant & maſquant,
" pour ainſi dire, la premiere, la rendent plus difficile à
" découvrir. Ceux qui ont quelque connoiſſance de ce
" qu'on appelle en Algebre la théorie des équations, ſavent
" par expérience la vérité de ce que nous venons de dire."

136. *Æquationum vero radices ſæpe impoſſibiles eſſe æquum eſt, ne caſus problematum qui ſæpe impoſſibiles ſunt exhibeant poſſibiles*, § 34 & 46.

Et hic quidem obſervandum eſt, quod in *Algebrâ* radix impoſſibilis eſt tantum nota ſeu ſymbolum operationis quæ in certo quodam caſu fieri nequit, ejuſque idcirco munus eſt *geneſin* quantitatis quæſitæ, & *limites* intra quos conſiſtere poteſt, indicare: *Geometria* vero radices impoſſibiles proprie non agnoſcit, ex eo quod ſolutio quæſtionis geometrice tantum exponi poteſt ubi ſit realis. Ut ſi rectæ & circuli interſectio determinanda eſſet, deſignantibus $p$ perpendiculum a centro in rectam demiſſum, $r$ radium ductum ad interſectionem, & $x$ diſtantiam interſectionis a $p$, habetur æquatio interſectionem definiens $x^2 = r^2 - p^2$, per *Eucl. Lib.* 1, *Prop.* 47. Erit autem interſectio poſſibilis ſi $p$ ſit minus quam $r$, ſin majus, fiet impoſſibilis ; & æquationis duæ radices quæ interſectiones duas determinant, nempe $x = \pm \sqrt{r^2 - p^2}$, debent eſſe perinde poſſibiles vel impoſſibiles ut rem ipſam vere experimant. Cæterum haſce radices geometrice tantum exponere fas eſt ubi interſectio eſt poſſibilis.

137. Aliquando fieri poteſt, velut in cubicarum *caſu irreducibili*, quod radices *reales* ſub *ſpecie impoſſibili* adumbrentur ; cæterum hoc fit ex eo quod conditiones in ſolutionem æquationis adſcitæ, ſibi in iſto caſu refragantur.

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## De numero radicum æquationis.

138. *Poteſt æquatio tot habere radices quot ſunt dimenſiones ejus, & non plures; ſi modo termini ejus ex ſimplicibus poteſtatibus integris conſtantur.*

Æquationem ſimplicem unam ſortiri radicem, quadraticam vero binas, ex Parte I. manifeſtum erat; unde in æquationibus dimenſionum ſuperiorum, ſimilem eſſe rationem radicum, per analogiam conjicere liceret. Cæterum numerus et natura radicum ex *generatione æquationis* optime intelligetur.

Ut ſi ſcire vellemus quomodo generetur æquatio cujus radices ſint 1, 2, 3, vel — 5, obſervando retrogradam compoſitionis viam, ſupponamus $x$ ambigue ſignificare numeros illos, ſeu eſſe $x = 1$, $x = 2$, $x = 3$, vel $x = -5$, adeoque per tranſpoſitionem $x - 1 = 0$, $x - 2 = 0$, $x - 3 = 0$, vel $x + 5 = 0$. Et multiplicando hæc in ſe, prodibit multiplicatione $x - 1$ in $x - 2$ hæc æquatio $x^2 - 3x + 2 = 0$, quæ duarum eſt dimenſionum ac duas habet radices 1 & 2; nam ſcriptâ 1 pro $x$ evadit factor prior $1 - 1 = 0$, vel ſcripto 2 pro $x$ evadit factor poſterior $2 - 2 = 0$, unde in alterutro caſu factum ſub utriſque nihil erit, ſcil. $1 - 3 + 2 = 0$, vel $4 - 6 + 2 = 0$. Porro, ductis in ſe tribus factoribus $x - 1$, $x - 2$, & $x - 3$, generatur $x^3 - 6x^2 + 11x - 6 = 0$, æquatio trium dimenſionum totidemque radicum 1, 2 vel 3; nam ſi $x$ ambigue ſignificat 1, 2 vel 3, evaneſcet primus, ſecundus aut tertius factor reſpective, at quolibet factore evaneſcente evaneſcet factum ſub tribus. Denique ductis



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in ſe quatuor factoribus $x - 1$, $x - 2$, $x - 3$. & $x + 5$ fit $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, ut ſupra, æquatio quatuor dimenſionum totidemque radicum 1, 2, 3, vel $-5$. Nam ſi $x$ ambigue, &c. Cum igitur hæc æquatio ex tot factoribus ſimplicibus generatur quot ſunt dimenſiones ejus, & factor quiſque ſimplex unam ſuppeditat radicem, ſequitur eam tot habere radices quot ſunt dimenſiones ejus. Non autem plures, cum præter hoſce numeros nullus eſt cujus ſubſtitutione factor aliquis evadit nihil; ubi vero factorum nullus nihil eſt, quod ſub omnibus continetur nihil eſſe non poteſt. Et ſimile eſt ratiocinium de omnibus æquationibus. Nam tali multiplicatione imaginari poſſumus omnes generari, quamvis factores ab invicem ſecernere ſolet eſſe difficillimum, & ipſum ſit quod æquationem reſolvere & radices extrahere: habitis enim radicibus habentur factores.

*Sed revocare gradus* ——
*Hic labor*, *hoc opus*.

139. Poſſunt quoque æquationes plurium dimenſionum, ex aliis inferiorum dimenſionum, illarum dimenſiones ſimul æquiparantium generari. Nam æquationem cubicam ex quadraticâ & ſimplici in ſe ductis; biquadraticam ex cubica & ſimplici vel ex duabus quadraticis in ſe ductis generari concipiamus; atque ita deinceps. Sic æquationis $x^4 - x^3 - 19x^2$ &c. factores ſunt $x^3 - 6x^2$ &c. et $x + 5 = 0$, vel $x^2 - 3x + 2 = 0$, & $x^2 + 2x - 15 = 0$. Itemque æquationis $x^3 - 1 = 0$, factores ſunt $x^2 + x + 1 = 0$, & $x - 1 = 0$. Factorum vero quadraticorum, &c. radices erunt radices æquationum.

140. Hanc generationem æquationum, in quadraticis ſaltem, ex principiis *Geometricis* inſuper confirmare & illuſtrare jam aggrediemur.

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Nam ad quadraticas puras quod attinet, ſi habentur duæ quantitates $x$ & $q$, quarum major ſit $x$, erit $x^2 - q^2 = (x + q)(x - q)$, per *Eucl. Lib.* 2. *Prop.* 5. At ſi $x$ fiat æqualis $q$, evaneſcet tum $x^2 - q^2$ tum $x - q$. Unde legitime concludamus æquationem algebraicam $x^2 - q^2 = 0$, gene[illegible]ſſe ex multiplicatione factorum ſimplicium [illegible] $x - q$, quorum alter poteſt eſſe nihil. Contra, [illegible] $q^2 = \frac{(x + q)^2 + (x - q)^2}{2}$, per *Eucl. Lib.* 2. *Prop.* 9. M[illegible] recte igitur concludamus conſtitui poſſe æquationem [illegible]lgebraicam $x^2 + q^2 = 0$, neutiquam evaneſcente $\frac{(x + q)^2 + (x - q)^2}{2}$; aut generari poſſe multiplicatione du[illegible]um factorum ſimplicium, cum ex ſemiſumma quadr[illegible]orum ſummæ & differentiæ $x$ & $q$ revera conflatur. E[illegible]o prodeunt hujus factores $x \pm q\sqrt{-1}$ impoſſibiles, ut a[illegible]um eſt.

Eadem quoque obtinent in quadraticis affectis. Nam ſi [illegible]ta quædam quantitas $p$ dividatur duas in partes inæquales quarum una ſit $x$, alteram deſignabit $p - x$, et deſignante $q$ rectangulum partium, erit $q = px - x^2$, per *Eucl. Lib.* 2. *Prop.* 3, et horum differentia nempe $x^2 - px + q$ æquatur nihilo. Unde legitime concludamus æquationem algebraicam $x^2 - px + q = 0$, generari poſſe multiplicatione factorum $x - \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$ & $x - \frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$, quorum alteruter erit nihil, ob $x - \frac{1}{2}p = \pm \sqrt{\frac{1}{4}p^2 - q}$; & quorum uterque erit poſſibilis, cum $q$ in hoc caſu eſt neceſſario minus quam $\frac{1}{4}p^2$, per *Eucl. Lib.* 2. *Prop.* 5. Contra, deſignantibus $x$ & $q$ latera trianguli angulum acutum comprehendentia, $p$ ſegmentum majoris lateris $x$ interceptum inter perpendiculum ab angulo oppoſito & angulum acutum, exponet $x^2 - 2px + q^2$ quadratum lateris reliqui angulo acuto oppoſiti, per *Eucl. Lib.* 2. *Prop.* 13.

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At hoc latus, ſalvo triangulo nunquam evaneſcere poteſt. Ergo æquationis $x^2 - 2px + q^2 = 0$, factores $x - p \pm \sqrt{p^2 - q^2}$ prodeunt impoſſibiles, cum $q^2$ in hoc caſu, eſt neceſſario majus quam $p^2$, per *Eucl. Lib.* 1. *Prop.* 47. Et ex hypotheſi impoſſibili concluſionem impoſſibilem colligi mirum eſſe non debet.

141. Uſum æquationum originalium $x - a = 0$, $x^2 - 20x - 9 = 0$, in compoſitionem æquationum ſuperiorum dimenſionum, qualem explicuimus, primus patefecit ingens ille algebræ hodiernæ repertor ac inſtaurator *Vieta* *; qui methodum generalem invenit pro educendis radicibus ex æquatione qualibet, *ubique*, ut ait, *obſervando retrogradam compoſitionis viam.* Hujuſque veſtigiis inſiſtentes *Harriott*, *Oughtred*, aliique tam noſtrates quam extranei quæcunque de hac re ſcriptis mandarunt, a *Vieta* deſumpta debent agnoſcere; judice *Halley*, *Phil. Tranſ. An.* 1687. No 190. *Art.* 2. *Montucla*, &c.

Ex generatione æquationum, regulam inſequentem in analyſi multis nominibus perutilem, atque uberrimam elicuit *Harriott*.

* *Vieta* vir fuit ingenioſa & profunda meditatione, cujus vi nihil illi inacceſſum in abſtruſioribus ſcientiis, nihil quod acumine mentis poſſet confici, difficile confectu fuit. Hujus tale dedit indicium. Bello inter Gallos Hiſpanoſque ingruente, literæ pleræque Hiſpanorum quibus præfectos ad Belgiam de conſiliorum ſuorum rationibus monebant, interceptæ, ob ingentem notarum multitudinem, (utebantur enim amplius 500, exoleto & incognito charactere exaratis) eos qui vulgo in his rebus induſtriam ſuam exercent prorſus eluſerunt. Itaque ad Vietam regis juſſu miſſæ ſunt, qui familiari ſibi in ſtudiis gravioribus meditatione illud totum expiſcatus eſt, & pleraſque deinceps alias maximi momenti, reperto ſemel arcano, nullo negotio interpretatus eſt. Quod res Hiſpanas totum biennium valde conturbavit. Qui, re tandem per literas viciſſim interceptas detectâ, *Magicis Artibus*, nam aliter fieri non potuiſſe, a Rege id factum, paſſim publicabant.

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## De coefficientibus & signis terminorum æquationis.

142. *Ubi coefficiens* primi *ſeu altiſſimi termini ſit unitas, coefficiens* ſecundi *termini æquationis, ſi ſignum ejus mutetur æqualis ſit aggregato omnium radicum ſub ſignis propriis; ea* tertii, *æqualis aggregato rectangulorum ſub ſingulis binis radicibus; ea* quarti *ſi ſignum ejus mutetur, æquales aggregato contentorum ſub ſingulis terni radicibus; ea* quinti, *æqualis aggregato contentorum ſub ſingulis quaternis, &c. et denique ea* ultimi *æqualis contento unico ſub omnibus radicibus.*

Aſſumamus $x = a$, $x = b$, $x = -c$, vel $x = d$, ſeu tranſponendo, $x - a = 0$, $x - b = 0$, $x + c = 0$, vel $x - d = 0$, et ex horum continua multiplicatione generemus æquationes, ut ſupra.

$$\begin{array}{l} x - a \\ \times\, \underline{x - b} \end{array}$$

$$= \left\{ x^2 \begin{array}{l} -a \\ -b \end{array} x + ab = 0. \right. \quad \text{Æquatio quadrat.}$$

$$\times\, \overline{x + c}$$

$$= \left\{ x^3 \begin{array}{l} -a \\ -b \\ +c \end{array} x^2 \begin{array}{l} +ab \\ -ac \\ -bc \end{array} x + abc = 0. \right. \quad \text{Æquatio cubica.}$$

$$\times\, \overline{x - d}$$

$$= \left\{ x^4 \begin{array}{l} -a \\ -b \\ +c \\ -d \end{array} x^3 \begin{array}{l} +ab \\ -ac \\ +ad \\ -bc \\ +bd \\ -cd \end{array} x^2 \begin{array}{l} +abc \\ -abd \\ +acd \\ +bcd \end{array} x - abcd = 0. \right. \quad \text{Æq. biquadrat.}$$

&c.

143. Ex harum æquationum inſpectione conſtat, quod coefficiens ſecundi termini ſub ſignis mutatis, nimirum $+ a + b$ in quadratica, $+ a + b - c$ in cubica, $+ a + b - c + d$ in biquadratica, eſt ſumma omnium radicum; coefficiens tertii $ab - ac + ad$, &c. ſumma rectangulorum ſub ſingulis binis; ea quarti ſub ſignis mutatis, $- abc + abd$, &c. ſumma contentorum ſub ſingulis ternis; atque ea ultimi, nimirum $+ ab$ in quadratica, $- abc$ in cubica, $- abcd$ in biquadratica, contentum unicum ſub omnibus. Et ſic univerſaliter: æquationis cujuſcunque $x^n \,.\, Px^{n-1} \,.\, Qx^{n-2} \,.\, Rx^{n-3} \,.\, Sx^{n-4}$ &c. $= 0$, coefficiens P, ſi ſignum ejus mutetur, æqualis ſit ſummæ radicum $a$, $b$, $c$, &c. ſub ſignis propriis; Q æqualis ſummæ rectangulorum ſub ſingulis binis $ab$, $ac$, &c. $bc$, $bd$, &c. $cd$, &c. R, ſi ſignum ejus mutetur, æqualis ſummæ contentorum ſub ſingulis ternis $abc$, $abd$, &c. $bcd$, $bce$, &c. $cde$, &c. S æqualis ſummæ contentorum ſub ſingulis quaternis $abcd$, &c. $bcde$, &c. $cdef$, &c. Et ſic in infinitum.

144. Hic obſervandum eſt, quod mutentur ſigna terminorum alternorum a ſecundo incipientium, quorum coefficientes ſunt aggregata contentorum ſub *impari* numero radicum ſignis mutatis; reliquorum vero terminorum non item. Nam in priore caſu, mutatis ſignis radicum numero *imparium*, mutantur ſigna ſingulorum contentorum, adeoque et aggregati contentorum in unoquoque termino; at in poſteriore caſu, ſive ſervatis ſive mutatis ſignis radicum numero *parium*, manent ſingulorum contentorum adeoque et aggregati ſigna. Sic $a \times b$ vel $- a \times - b = + ab$. At $a \times b \times - c = - abc$, & $- a \times - b \times c = + abc$. Et ſic deinceps.

Cor. I. Hinc cujuſlibet termini æquationis, puta termini $m^{\text{t}}$, ſive inter terminos pares ſive impares inveniatur, coefficiens æqualis erit aggregato contentorum ſub ſingulis $\overline{m-1}$ radicibus, cum ſignis mutatis. Eritque numerus terminorum æquationis omnes ſuos terminos habentis, uno plus quam numerus radicum ſeu dimenſionum.

Cor. II. Si omnes radices æquationis ſint affirmativæ, termini alterni a ſecundo incipientes erunt negativi, cæteri affirmativi; ſin omnes radices ſint negativæ, termini omnes erunt affirmativi.

Cor. III. Si radices ſint partim affirmativæ partim negativæ, præpollentibus affirmativis erit ſecundus terminus negativus, & vice verſa; æquipollentibus vero affirmativis & negativis, evaneſcet. Itemque, præpollentibus rectangulis affirmativis &c. erit tertius terminus affirmativus, & vice verſa; æquipollentibus vero evaneſcet. Et ſic deinceps.

Cor. IV. Si ultimus terminus æquationes deſit, ex radicibus ejus una erit nihil, aliter enim non evaneſceret contentum ſub omnibus. Si penultimus quoque deſit, ex radicibus duæ evaneſcent, aliter enim non evaneſcerent ſingula contenta ex quibus ſimul ſumptis conficitur penultimi coefficiens. Si tres poſtremi termini deſint, tres ex radicibus evaneſcent. Et ſic deinceps.

Cor. V. Æquationis cujuſcunque terminos nec fractos nec ſurdos habentis nulla radix fracta eſſe poteſt. Nam æquationis $x^n - Px^{n-1} + Qx^{n-2} - \&c. + U = 0$, radix ſit fractio $\frac{a}{b}$, ad minimos terminos ejus reducta; &

ſubſtituendo $\frac{a}{b}$ pro $x$, prodibit $\frac{a^n}{b^n} - \frac{Pa^{n-1}}{b^{n-1}} + \frac{Qa^{n-2}}{b^{n-2}}$ — &c. + U = o; deinde multiplicando omnes terminos per $b^{n-1}$ & tranſponendo, erit $\frac{a^n}{b} = Pa^{n-1} - Qa^{n-2}b$ + &c. — $Ub^{n-1} = o$, cujus membrum poſterius eſt integer, adeoque & prius $\frac{a^n}{b}$. Unde erit $b$ aut unitas, aut ex diviſoribus $a^n$. At in priore caſu non foret $\frac{a}{b}$ fractio, nec ad minimos terminos reducta in poſteriore caſu; contra hypotheſin. Poſſunt tamen radices eſſe ſurdæ. Sic æquationis $x^2 - x - 1 = o$, radices ſunt $\frac{1+\sqrt{5}}{2}$ & $\frac{1-\sqrt{5}}{2}$.

Cor. VI. Æquationis cujuſcunque terminos nec fractos nec ſurdos habentis omnes radices non ſurdæ & radicum binarum rectangula, ternarumque aut plurium contenta erunt aliqui ex diviſoribus *integris* ultimi termini; atque adeo ubi conſtiterit nullum ultimi termini diviſorem eſſe aut radicem æquationis aut duarum radicum rectangulum pluriumve contentum, ſimul conſtabit nullam eſſe radicem radicumve rectangulum aut contentum niſi quod ſit ſurdum.

Cor. VII. Æquationis cujuſcunque terminos nullos impoſſibiles habentis radices impoſſibiles, ſiquæ ſint, erunt 1. Numero pares. Nam ultimus terminus utpote contentum ſub omnibus radicibus impoſſibilitate non vacaret, ſi compoſitionem ejus ingrederetur numerus impar factorum im-

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poſſibilium. 2. Paris cujuſque partes impoſſibiles eædem erunt & contrariis ſignis affectæ. Aliter enim hæ partes per additionem non evaneſcerent ex termino ſecundo æquationis. 3. Paris cujuſque partes reales, ſi quæ ſint, erunt eædem & iiſdem ſignis affectæ. Nam rectangula & contenta impoſſibilia ſub ſingulis binis terniſve, &c. radicibus, ex reliquis terminis haud exularent niſi hæc contenta forent eadem & contrariis ſignis affecta; ſed factores horum impoſſibiles per *part. noviſſ.* ſunt iidem & contrariis ſignis affecti; ergo factores reales neceſſario erunt iidem & iiſdem ſignis affecti.

Neceſſe eſt igitur ut hujuſmodi æquationum radices impoſſibiles, ſi quæ ſint, induant vel formam $\pm a + b\sqrt{-1}$ ubi ex parti reali & impoſſibili conflantur, vel $\pm b\sqrt{-1}$ ubi nullam ſortiuntur partem realem. Ex. gr.

Aſſumamus $x = a + b\sqrt{-1}$, $x = a - b\sqrt{-1}$, $x = +b\sqrt{-1}$, vel $x = -b\sqrt{-1}$. Et ex factorum $x - a \mp b\sqrt{-1}$ vel $x \mp b\sqrt{-1} = 0$, continua multiplicatione generemus æquationes; eritque

$$\begin{array}{l} \phantom{\times}\ x - a - b\sqrt{-1} \\ \times\ x - a + b\sqrt{-1} \\ \hline \end{array}$$

$$x^2 - 2ax \begin{matrix} +a^2 \\ +b^2 \end{matrix} = 0.$$ Æquatio quadratica.

$$\begin{array}{l} \times\ x - b\sqrt{-1} \\ \hline \end{array}$$

$$x^3 \begin{matrix} -2a \\ -b\sqrt{-1} \end{matrix}\, x^2 \begin{matrix} +a^2 \\ +b^2 \\ +2ab\sqrt{-1} \end{matrix}\, x \begin{matrix} -a^2b\sqrt{-1} \\ -b^3\sqrt{-1} \end{matrix} = 0.$$ Æq. cub.

$$\begin{array}{l} \times\ x + b\sqrt{-1} \\ \hline \end{array}$$

$$x^4 - 2ax^3 \begin{matrix} +a^2 \\ +2b^2 \end{matrix}\, x^2 - 2ab^2x \begin{matrix} +a^2b^2 \\ +b^4 \end{matrix} = 0.$$ Æq. biquadratica.



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Ubi obſervandum, quod in æquationibus parium dimenſionum pelluntur quantitates impoſſibiles ex ſecundo termino per additionem ſolam, ex ultimo termino per multiplicationem ſolam, & ex terminis intermediis per additionem & multiplicationem conjunctim; contra, in terminis æquationum imparium dimenſionum ſemper hærent quantitates impoſſibiles.

Cor. VIII. Æquationis cujuſcunque dimenſionis imparis omnes ſuos terminos reales habentis erit una ſaltem radix realis, cum numerus radicum impoſſibilium ſemper ſit par, *per cor. præced.*

Cor. IX. Deſignante P, ſi ſignum ejus mutetur, ſummam radicum $a$, $b$, $c$, &c, ſub ſignis propriis, mutando inſuper ſigna alternorum coefficientium a ſecundo incipientium R, T, &c. facile invenire poſſumus ſummam quadratorum, cuborum, biquadratorum, &c. ex ſingulis radicibus, ope formularum § 79. Ut in æquatione $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, ubi coefficiens ſecundi termini eſt $-1$, tertii $-19$, quarti $+49$, quinti $-30$; ponendum erit $+1 = P$, $-19 = Q$, $-49 = R$, $-30 = S$. Et inde orientur

$A = P = 1$, *ſumma radicum.*
$B = PA - 2Q = 1 + 38 = 39$, *ſumma quadratorum.*
$C = PB - QA + 3R = 39 + 19 - 147 = -89$, *ſumma cuborum.*
$D = PC - QB + RA - 4S = -89 + 741 - 49 + 120 = 723$, *ſum. biq.*

Nimirum æquationis illius radices ſunt 1, 2, 3 & $-5$, et harum ſumma $1 + 2 + 3 - 5 = 1$, ſumma quadratorum $1 + 4 + 9 + 25$ eſt 39, ſumma cuborum $1 + 8 + 27 - 125$ eſt $-89$, & ſumma biquadratorum $1 + 16 + 81 + 625$ eſt 723.

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Paulo aliter, et commodius quidem, calculum inſtituit *Newtonus*; mutat enim ſigna *omnium* coefficientium, quo evitetur confuſio (ut videtur) ſi qua forte inter operandum ſuboriatur, mutandis tantum ſignis coefficientium alternarum P, R, T, &c. Dein, quo compenſetur mutatio ſignorum reliquarum Q, S, V, &c. mutat quoque ſigna terminorum formulis in ſingulis negativorum quos nempe ſubeunt hæ coefficientes Q, S, V, &c. quo pacto *omnes* formularum termini prodeunt affirmativi; ſcilicet, $B = PA + 2Q$, $C = PB + QA + 3R$, &c.

Cor. X. Facile determinari poteſt numerus contentorum ſub ſingulis binis, ternis, pluribuſve radicibus, in unoquoque coefficiente P, Q, R, &c. ope Theorematis Binomii. Nam ſi omnes radices *a*, *b*, *c*, &c. ſint inter ſe *æquales*, ſingulas deſignabit *a*, eritque $(x - a)^n = x^n - nax^{n-1} + \frac{n}{1} \cdot \frac{n-1}{2} a^2x^{n-2} - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^3x^{n-3} + \&c. = 0$, adeoque P coefficiens ſecundi termini $= na$, $Q = n \cdot \frac{n-1}{2} a^2$, $R = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^3$ &c. Exponet igitur $n$ ſummam radicum $a$; $n \cdot \frac{n-1}{2}$ ſummam rectangulorum ſub ſingulis binis $aa$; $\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$ ſummam contentorum ſub ſingulis ternis $aaa$, &c. Verum hoc quoque ita ſe habebit, ſi radices *a*, *b*, *c*, &c. ſint *inæquales*; cum hæc conditio nec mutabit numerum radicum ſingula contenta componentium, nec ipſorum contentorum numerum in unoquoque termino æquationis. Ex. gr. Sit numerus radicum $n = 4$; erit numerus con-

tentorum ſub ſingulis binis $\frac{4\cdot 3}{1\cdot 2}=6$; numerus contentorum ſub ſingulis ternis $\frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}=4$; ſub ſingulis quaternis $\frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}=1$; ut in æquatione biquadratica ſupra allatâ. Eſtque hæc regula in inveſtigandis plurium quantitatum *combinationibus* diverſis admodum utilis.

Cor. XI. Æquationis cujuſcunque $x^{n} . Px^{n-1} . Qx^{n-2}$ &c. $=0$, omnes radices reales habentis, quadratum coefficientis termini cujuſque medii majus erit rectangulo coefficientium terminorum hinc inde conſiſtentium, ſi modo hoc rectangulum affirmativum ſit. Nam

I. Si radices ſint eædem & iiſdem ſignis affectæ; deſignabit $na$, $n . \frac{n-1}{2} a^{2}$, $n . \frac{n-1}{2} . \frac{n-2}{3} a^{3}$, &c. ſeriem coefficientium P, Q, R, &c. reſpective; per *cor. præced.* atque ideo, rationem trium quarumvis coefficientium ſucceſſivarum, ſi dividantur ſigillatim per primam ex his, exponet $1$, $\frac{n-m}{m+1} a$, $\frac{n-m}{m+1} . \frac{n-\overline{m+1}}{m+2} a^{2}$. (ubi *m* deſignat integrum quemvis minorem *n*.) Sed $\left|\frac{n-m}{m+1} a\right|^{2}$ ſeu $\frac{n-m}{m+1} . \frac{n-m}{m+1} a^{2}$ majus eſt quam $1 \times \frac{n-m}{m+1} . \frac{n-\overline{m+1}}{m+2} a^{2}$, ob $\frac{n-m}{m+1}$ majus quam $\frac{n-\overline{m+1}}{m+2}$. Ergo, quadratum &c. majus eſt rectangulo, &c.

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II. Si radices ſint diverſæ at iiſdem ſignis affectæ, ſumptis tribus quibuſvis coefficientibus ſucceſſivis, puta R, S, T, erunt

*Formæ terminorum in* $S^2$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1. | $a^2b^2c^2d^2$ | + $a^2b^2c^2e^2$ | + &c. | + $b^2c^2d^2e^2$ | + &c. | [C4] |
| 2. | $a^2b^2c^2de$ | + $a^2b^2c^2df$ | + &c. | + $b^2c^2d^2ef$ | + &c. | [αC5] |
| 3. | $a^2b^2cdef$ | + $a^2b^2cdeg$ | + &c. | + $b^2c^2defg$ | + &c. | [βC6] |
| 4. | $a^2bcdefg$ | + $a^2bcdefh$ | + &c. | + $b^2cdefgh$ | + &c. | [γC7] |
| 5. | $abcdefgh$ | + $abcdefgk$ | + &c. | + $bcdefghk$ | + &c. | [δC8] |

*De coefficientibus formarum in* $S^2$.

1. $a^2b^2c^2d^2$ ſeu $abcd \times abcd$. Cum ex quatuor radicibus quatuor fiunt quadrata, erit numerus combinationum $\frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} = 1$; adeoque quiſque terminus hujus formæ ſemel tantum occurrit: & totum numerum terminorum ejus exponet C4 ſeu numerus combinationum ſub ſingulis radicibus quaternis in æquatione ($n$) radicum, $= n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$. per *cor. præced.*

2. $a^2b^2c^2de$ ſeu $abcd \times abce$. Cum ex quinque radicibus ternæ quævis fiunt quadrata, erit numerus combinationum $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}$; quarum quæque bis occurrit, cum binæ reliquæ radices modis $\frac{2}{1}$ in factoribus diſponi poſſunt, ſcilicet

$abcd \times abce$ vel $abce \times abcd$: erit igitur coefficiens termini cujuſque $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \times \frac{2}{1} =$ puta $\alpha$, & totus numerus terminorum hujus formæ, $\alpha C5$.

3. $a^2b^2cdef$ ſeu $abcd \times abef$. Cum ex ſex radicibus binæ quæque fiunt quadrata, prodit numerus combinationum $\frac{6 \cdot 5}{1 \cdot 2}$; & quatuor reliquæ modis $\frac{4 \cdot 3}{1 \cdot 2}$ in factoribus diſponi poſſunt: erit igitur coefficiens termini cujuſque hujus formæ $\frac{6 \cdot 5}{1 \cdot 2} \times \frac{4 \cdot 3}{1 \cdot 2} = \beta$, & totus numerus terminorum, $\beta C6$. Similiter,

4. $a^2bcdefg$ ſeu $abcd \times aefg$. Erit coefficiens termini cujuſque hujus formæ $\frac{7}{1} \times \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = \gamma$, & totus numerus terminorum $\gamma C7$.

5. $abcdefgh$ ſeu $abcd \times efgh$. Erit coefficiens termini cujuſque hujus formæ $\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = \delta$, & totus numerus terminorum $\delta C8$.

### *Formæ terminorum in* RT.

1. $a^2b^2c^2de + a^2b^2c^2df +$ &c. $+ b^2c^2d^2ef +$ &c. $[\frac{1}{2}\alpha C5]$
2. $a^2b^2cdef + a^2b^2cdeg +$ &c. $+ b^2c^2defg +$ &c. $[\frac{2}{3}\beta C6]$
3. $a^2bcdefg + a^2bcdefh +$ &c. $+ b^2cdefgh +$ &c. $[\frac{3}{4}\gamma C7]$
4. $abcdefgh + abcdefgk +$ &c. $+ bcdefghk +$ &c. $[\frac{4}{5}\delta C8]$

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Ubi obſervandum, quod omnes formas in $S^2$ præter primam ſortitur RT.

*De coefficientibus formarum in* RT.

1. $a^2b^2c^2de$ ſeu $abc \times abcde$. erit coefficiens $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = \frac{1}{2}\alpha$, adeoque totus numerus terminorum, $\frac{1}{2}\alpha C5$.

2. $a^2b^2cdef$ ſeu $abc \times abdef$. coeff. erit $\frac{6 \cdot 5}{1 \cdot 2} \times \frac{4}{1} = \frac{2}{3}\beta$, adeoq. &c.

3. $a^2bcdefg$ ſeu $abc \times adefg$. coeff. erit $\frac{7}{1} \times \frac{6 \cdot 5}{1 \cdot 2} = \frac{3}{4}\gamma$, adeoq. &c.

4. $abcdefgh$ ſeu $abc \times defgh$. coeff. erit $\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = \frac{4}{5}\delta$, adeoq. &c.

Hinc erit

$$\begin{array}{l} S^2 = C4 + \alpha C5 + \beta C6 + \gamma C7 + \delta C8 \\ RT = \quad + \frac{1}{2}\alpha C5 + \frac{2}{3}\beta C6 + \frac{3}{4}\gamma C7 + \frac{4}{5}\delta C8 \\ \hline S^2 - RT = C4 + \frac{1}{2}\alpha C5 + \frac{1}{3}\beta C6 + \frac{1}{4}\gamma C7 + \frac{1}{5}\delta C8. \text{ affirm.} \end{array}$$

Et ſic univerſaliter, deſignante Y ſummam productorum ſub ſingulis ($m$) radicibus in æquatione ($n$) dimenſionum, & deſignantibus $\alpha$, $\beta$, $\gamma$, $\delta$, &c. correſpondentes coefficientes numerales quantitatum $C\overline{m+1}$, $C\overline{m+2}$, $C\overline{m+3}$, &c. erit, $Y^2 - XZ = Cm + \frac{1}{2}\alpha C\overline{m+1} + \frac{1}{3}\beta C\overline{m+2} + \frac{1}{4}\gamma C\overline{m+3}$ &c. uſque ad $m+1$ terminos. Ergo, &c.

III. Si radices ſint diverſæ & diverſis ſignis affectæ; ob ingentem æquationum varietatem, hunc caſum univerſaliter demonſtrare difficillimum. Veritas tamen ejus ita per inductionem conſtabit.



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In æquatione quadratica $x^2 \pm px + q = 0$, ubi radices ſunt reales, erit $\frac{1}{4}p^2$ æquale aut majus quam $q$, ſive $p^2$ æquale aut majus quam $4q$, per § 29. Adeoque *a fortiori*, $p^2$ majus quam $q$.

Et multiplicando quadraticam $x^2 + px + q = 0$ per $x - c = 0$, generabitur æquatio cubica $x^3 + (p - c)x^2 + (q - pc)\,x - qc = 0$, & deſignante $m$ numerum æqualem aut majorem quaternario, erit

$$\begin{aligned} P^2 &= p^2 - 2pc + c^2 = mq - 2pc + c^2. \\ Q &= \phantom{p^2 - 2pc + c^2 = m} q - pc \\ \hline \text{Adeoque } P^2 - Q &= \overline{m-1}\,q - pc + c^2. \text{ affir.} \end{aligned}$$

Ubi $q - pc$ eſt affirmativum, per hypoth. et $\overline{m-1}\,q$ eſt majus quam $q$, adeoque $\overline{m-1}\,q - pc$ eſt affirmativum, & *a fortiori*, $\overline{m-1}\,q - pc + c^2$ affirmativum.

Erit quoque

$$\begin{aligned} Q^2 &= p^2c^2 - 2pqc + q^2 = mqc^2 - 2pqc + q^2 \\ PR &= \phantom{p^2c^2 - 2pqc + q^2 = m} qc^2 - pqc \\ \hline Q^2 - PR &= \overline{m-1}\,qc^2 - pqc + q^2 \text{ affirm.} \end{aligned}$$

Et ſimiliter probare licet in æquatione cubica $(x^2 - px + q)(x + c) = 0$, &c.

Multiplicando autem quadraticam $x^2 + px + q = 0$ per quadraticam $x^2 - gx + b = 0$, generabitur biquadratica $x^4 + (p - g)\,x^3 + (q - pg + b)\,x^2 + (pb - qg)\,x + qb = 0$, adeoque ob $g^2 =$ vel $> 4b$, $=$ puta $nb$, erit

$$P^2 = p^2 - 2pg + g^2 = mq - 2pg + nh$$
$$Q = q - pg + h$$

Adeoq. $P^2 - Q = \overline{m-1}\, q - pg + \overline{n-1}\, h$ affir.

Itemque,

$$Q^2 = p^2g^2 - 2pg(q+h) + (q+h)^2 = (mn+2)qh - 2pg(q+h) + q^2 + h^2$$
$$PR = p^2h - pg(q+h) + qg^2 = (m+n)qh - pg(q+h)$$

Adeoq. $Q^2 - PR = (\overline{mn+2} - \overline{m+n})\, qh - pg(q+h) + q^2 + h^2$.

affirmativum nempe, ob $\overline{mn+2} - \overline{m+n}$ majus quam $\overline{m+n}$, cum $m$ & $n$ ſint quaternario majores aut ſaltem æquales, reſpective. Itemque,

$$R^2 = p^2h^2 - 2pqhg + q^2g^2 = mqh^2 - 2pqgh + nq^2h$$
$$QS = qh^2 - pqgh + q^2h$$

$R^2 - QS = \overline{m-1}\, qh^2 - pqgh + \overline{n-1}\, q^2h$

affirmativum.

Et multiplicando biquadraticam $x^4 + (p-g)\, x^3 +$ &c. $= 0$ per $x - e = 0$, in æquatione reſultante $x^5 + (p - g - e)\, x^4 + (q - pg + h - pc + cg)\, x^3 +$ &c. $= 0$, erit

$$P^2 = p^2 - 2pg - 2pe + 2ge + g^2 + e^2 = mq + 2 \times \overline{-pg - pe + ge} + nh + e^2$$
$$Q = q - pg - pe + ge + h = q + 1 \times \overline{-pg - pe + ge} + h$$

$P^2 - Q = \overline{m-1}\, q + 1 \times \overline{-pg - pe + ge} + \overline{n-1}\, h + e^2$

affirmativum. Atque ſic deinceps.



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Cor. XII. Hinc in æquatione quacunque ſi quadratum coefficientis termini cujuſvis medii ſit æquale aut minus rectangulo coefficientium terminorum hinc inde conſiſtentium, ubi hoc rectangulum ſit affirmativum; indicio eſt, æquationem radices impoſſibiles ſortiri. Veruntamen non valet *converſa*, ubi enim radices ſunt impoſſibiles, poteſt quadratum &c. majus eſſe hoc rectangulo. Sic in æquatione quadraticâ $x^2 \pm px + q = 0$, radices erunt impoſſibiles ubi $\frac{1}{4}p^2$ ſit minus quam $q$, at ſalvâ hac conditione, poteſt eſſe $p^2$ majus, æquale, vel minus quam $q$.

Hinc quoque defectus termini cujuſvis medii, ſi modo termini hinc inde conſiſtentes iiſdem ſignis afficiantur, eſt nota radicum impoſſibilium. Nam quadratum, &c. in hoc caſu evaneſcit, atque ideo minus prodit rectangulo, &c.

## De numero radicum affirmativarum & negativarum æquationis.

145. *Æquatio* cujuſcunque *dimenſionis cujus ultimus terminus eſt negativus, unam ſaltem habet radicem realem* & affirmativam.

Nam in æquatione $x^n \cdot x^{n-1}$ &c. $- U = 0$, ſi $x$ ſit nihil, evaneſcentibus omnibus terminis præter ultimum, erit valor reſultans negativus; contra; creſcente $x$ a nihilo ad magnitudinem infinitam uſque, terminis affirmativis tandem exſuperantibus negativos, valor reſultans erit affirmativus: ergo iſte valor affirmativus $x$ qui inter creſcendum efficiet omnes terminos ſe deſtruere, erit radix.

146. *Æquatio quæcunque dimenſionis* imparis *cujus ultimus terminus eſt negativus, unam ſaltem habet radicem realem* & affirmativam, *per* præced.

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—*Quod ſi ultimus terminus ſit affirmativus, unam ſaltem habet radicem realem &* negativam.

Subſtituendo enim $-x$ pro $+x$, primus terminus evadet negativus, & terminis omnibus novæ æquationis ad alteram partem tranſlatis, quo terminus altiſſimus evadat affirmativus, fiet ultimus terminus negativus, nova æquatio igitur ſortietur unam radicem realem & affirmativam, & proinde data æquatio unam realem & negativam.

147. *Æquatio quæcunque dimenſionis* paris *cujus ultimus terminus eſt negativus, neceſſario habet duas radices reales, unam* affirmativam, *alteram* negativam.

Habet enim unam radicem affirmativam, per § 145. Et ſi ſubſtituatur $-x$ pro $+x$, cum terminus altiſſimus (utpote dimenſionis paris) ſit affirmativus, nova æquatio habebit unam radicem realem & affirmativam, adeoque data æquatio alteram quoque radicem realem & negativam.

148. *Æquationis cujuſcunque unam tantum mutationem ſignorum* de $+$ in $-$ *habentis, unica erit radix realis* affirmativa.

Nam quoniam ultimus terminus in hoc caſu, neceſſario erit negativus, æquatio unam habebit radicem realem affirmativam, per § 145.

Quod ſi X deſignet ſummam terminorum affirmativorum, & Y ſummam negativorum, fiet æquatio $X - Y = 0$. Et quoniam per hypotheſin, eſt una tantum mutatio ſignorum, manifeſtum eſt, quod *minima* poteſtas $x$ in polynomio X, puta $x^m$, neceſſario tranſcendet *omnem* poteſtatem $x$ in polynomio Y. Adeoque dividendo per $x^m$, quan-

titas $\frac{X}{x^m}$ continebit ſolummodo poteſtates affirmativas $x$, & $\frac{Y}{x^m}$ poteſtates negativas $x$. Unde ſequitur, quod creſcente vel decreſcente $x$, perinde augebitur vel minuetur valor $\frac{X}{x^m}$; niſi X terminum unicum, nempe altiſſimum contineat, quo in caſu, non variabitur $\frac{X}{x^m}$. Contra, creſcente $x$, neceſſario minuetur valor $\frac{Y}{x^m}$, decreſcente vero $x$, augebitur. Sit jam $a$ radix realis affirmativa æquationis, eritque $x = a$, $X = Y$, itemque $\frac{X}{x^m} = \frac{Y}{x^m}$. Adeoque ſubſtituendo pro $x$ numeros majores quam $a$, erit ſemper $\frac{X}{x^m} > \frac{Y}{x^m}$ adeoque $\frac{X}{x^m} - \frac{Y}{x^m}$ ſeu $X - Y$ affirmativum; & ſubſtituendo pro $x$ numeros minores quam $a$, erit ſemper $\frac{X}{x^m} < \frac{Y}{x^m}$ adeoque $X - Y$ negativum. Ergo impoſſibile eſt æquationem ſortiri radicem majorem minoremve quam $a$ *.

* Vide *Diſſert.* Sur la reſolution des æquations numeriques, par *M. De la Grange, Berlin. Acad.* an. 1769, p. 314, &c.

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149. *Æquationis nullas radices impoſſibiles habentis tot ſunt radices* affirmativæ *quot ſignorum in continua ſerie* mutationes *de* + *in* —, & *de* — *in* +; *cæteræ* negativæ *ſunt.*

Hanc regulam in æquationum tranſmutationibus & limitibus inveſtigandis perutilem, tentando & inductione autorem inveniſſe, ſive *Harriott* fuit, ſive ut pleriſque placet, *Carteſius* *, veriſimile eſt. Demonſtrationem ejus, ex generatione æquationum, ita colligere licet.

In æquatione ſimplici $x \pm a = 0$ eſt una variatio ſignorum *de* + *in* — ubi radix eſt affirmativa, & una ſucceſſio *de* + *in* + ubi radix eſt negativa.

Et ſi multiplicetur $x + a = 0$, per $x - b = 0$, in æquatione quadr. reſultante $x^2 \begin{matrix} +a \\ -b \end{matrix} x - ab = 0$, cujus una radix eſt affirmativa & altera negativa, prodit ſeries ſignorum +, ±, —, prout $a$ ſit majus, æquale aut minus quam $b$, ſcilicet, vel +, +, —, vel +, —, —; adeoque in utroque caſu, una variatio ſignorum *de* + *in* —, & una ſucceſſio, vel *de* + *in* +, vel *de* — *in* —.

His veſtigiis inſiſtendo demonſtrare poſſimus regulam in æquationibus cubicis, ut facit *McLaurin, Algeb. Part.* 2, § 20. Verùm methodum eandem proſequi in æquationi-

* Regulam ſic effert *Carteſius, Geom. Lib.* 3, *p.* 70. " Ex quibus etiam cognoſcitur quot veræ & quot falſæ radices in unaquaque æquatione haberi poſſunt. Nimirum, tot in ea veras haberi poſſe, quot variationes reperiuntur ſignorum + & —; & tot falſas quot vicibus ibidem deprehendentur duo ſigna + vel duo ſigna —, quæ ſe invicem ſequuntur." Contenderunt nonnulli *Carteſio* latuiſſe limitationem regulæ ad æquationes quæ omnes radices habent reales; at minus recte, ut ex ipſius verbis colligere licet, *ex quibus*, nimirum ex generatione æquationum radices reales habentium, de quibus paulo ante egerat; itemque ex eo quod non ſimpliciter adhibet *haberi*, at cum reſtrictione, *haberi poſſe*, nimirum in caſu prædicto.

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bus altiorum dimenſionum, propter ambiguitatem ſignorum terminorum ſucceſſivorum, impoſſibile foret. Demonſtratio autem fiet univerſalis ope Theorematis inſequentis, quod ex *Meditationibus Algebraicis* D. *Waring*, p. 74. expreſſimus.

150. *Ubi æquationis radices nullæ impoſſibiles ſunt, ſi multiplicetur æquatio per* x — p = o, *ut una adhuc radix affirmativa addatur prioribus, augebitur unitate numerus mutationum ſignorum in nova æquatione; quod ſi multiplicatio fiat per* x + p = o, *ut una adhuc radix negativa addatur prioribus, idem erit numerus mutationum ſignorum in nova ac in data æquatione.*

$$\text{Sit } x^{n} - Px^{n-1} - Qx^{n-2} + Rx^{n-3} + Sx^{n-4} + Tx^{n-5} + \&c. = 0$$
$$\times x - p$$

$$x^{n+1} \begin{matrix} -P \\ -p \end{matrix} x^{n} \begin{matrix} -Q \\ +pP \end{matrix} x^{n-1} \begin{matrix} +R \\ +pQ \end{matrix} x^{n-2} \begin{matrix} +S \\ -pR \end{matrix} x^{n-3} \begin{matrix} +T \\ -pS \end{matrix} x^{n-4} \pm \&c. = 0.$$

Hic obſervandum eſt, quod binæ partes coefficientis termini cujuſque novæ æquationis iiſdem afficientur ſignis, ubi termini imminentis & præcedentis in data æquatione ſigna ſunt diverſa; contrariis vero afficientur ſignis, ubi termini imminentis & præcedentis ſigna ſunt eadem. Sic coefficiens ſecundi termini novæ æquationis eſt $-P-p$, exiſtente ſigno ſecundi termini datæ æquationis negativo, primi autem affirmativo; at coefficiens tertii termini eſt $-Q+pP$, negativum nempe, inſtar termini imminentis, ſi Q ſit majus quam $pP$, affirmativum ſi ſecus.

Unde, ſi trium pluriumve terminorum ſucceſſivorum datæ æquationis ſigna ſint eadem, erunt duorum pluriumve terminorum ſucceſſivorum novæ æquationis ſigna ambigua.

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Quod ſi talium cujuſlibet ſignum diverſum prodeat a ſigno termini imminentis datæ æquationis, termini proxime inſequentis quoque ſignum diverſum erit a ſigno termini huic imminentis. *Ex. gr.* Sit terminus quintus novæ æquationis $(S - pR)\, x^{n-3}$ negativus,

| | | | |
|---|---|---|---|
| Erit - - - - - - | S | minus quam | $pR$ |
| Sed per *Cor.* XI, eſt | RT | minus quam | $S^2$ |
| Adeoque - - - | RST | minus quam | $pRS^2$ |
| Sive - - - - | T | minus quam | $pS$. |

Et proinde terminus ſextus $(T - pS)\, x^{n-4}$ negativus erit. Et ſic deinceps. Erit igitur ſeries ſignorum

datæ æquat. $+, -, -, +, +, +, +,$ &c.
novæ æquat. $+, -, \mp, +, -, -, -,$ &c.

Nimirum a termino primo ad quartum, idem eſt numerus variationum in utriſque, idque ſive tertius terminus novæ æquationis ſit affirmativus ſive negativus; a quarto verò ad quintum in nova æquatione, accedit una variatio quæ non eſt in data; in reliquis terminis idem prodit numerus ſucceſſionum in utriſque. Ergo, &c.

Simili ratiocinio conſtat, quod ſi multiplicetur æquatio $x^n - Px^{n-1} -$ &c. $= 0$ per $x + p = 0$, idem reſultabit numerus mutationum ſignorum in nova ac in data æquatione.

Sic multiplicando quadraticam $x^2 + px + q = 0$ per $x - c = 0$, in æquatione cubica reſultante $x^3 + (p - c)$



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$x^2 + (q - pc)\, x - qc = 0$, cujus una radix eſt affirmativa, reliquæ negativæ; ſit $p$ majus quam $c$, poteſt eſſe $q$ majus vel minus quam $pc$, adeoque ſeries ſignorum $+$, $+$, $\pm$, $-$. Quod ſi $p$ ſit minus quam $c$, erit quoque $q$ minus quam $pc$, & ſeries ſignorum $+$, $-$, $-$, $-$. Nimirum in utroque caſu una variatio ſignorum & duæ ſucceſſiones, ut æquum eſt. Atque ita in æquationibus altiorum dimenſionum regulam demonſtrare licet.

151. Ubi vero radices aliquæ impoſſibiles ſunt, non valet regula *Carteſiana*, niſi quatenus impoſſibiles illæ quæ nec negativæ ſunt nec affirmativæ pro *ambiguis* habeantur. Ut in æquatione $x^3 + qx - r = 0$, interponendo $\pm\, 0x^2$ quo compenſetur defectus ſecundi termini, erit ſeries ſignorum vel $+$, $+$, $+$, $-$, vel $+$, $-$, $+$, $-$; nimirum in priore caſu, ſi ſigna ſpectes, una radix affirmativa & duæ negativæ, in poſteriore, tres affirmativæ. Sunt ergo duæ *impoſſibiles*, quæ pro ambiguitate ſua, priore caſu negativæ, poſteriore affirmativæ eſſe videntur.

Aliquando augendo numerum radicum æquationis detegentur radices impoſſibiles, ſi quæ in ea deliteſcunt. Sic in æquatione $x^3 + px^2 + 3p^2x - q = 0$, ſigna indicant radicem unam eſſe affirmativam, & duas negativas; multiplicando autem per $x - 2p = 0$, prodit $x^4 - px^3 + p^2x^2$ $(-6p^3 - q)\, x + 2pq = 0$, quæ habere deberet duas affirmativas ac duas negativas radices, habet tamen ſi mutationum ſignorum ſpectes, affirmativas quatuor; ſunt ergo duæ impoſſibiles. Cæterum in his & talibus exemplis radices impoſſibiles exiſtere, ex Cor. XII, ſtatim innoteſcit.

152. Regulam ſatis facilem tradit *Newtonus* pro inveniendo numero radicum impoſſibilium, at minus accuratam,

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quam idcirco prætereundam cenſui. De hac tale profert judicium *Waring*, in hiſce verſatiſſimus, *Med. Algeb.* p. 47. "Hæc methodus in quadraticis æquationibus verum præ-" "bet numerum impoſſibilium radicum; in cubicis autem" "probabilitas inveniendi impoſſibiles radices non videtur" "majorem habere rationem ad probabilitatem fallendi" "quam 2 ad 1. In æquationibus autem multo ſuperiorum" "dimenſionum verum impoſſibilium radicum numerum" "perraro deteget."

Sic in æquatione $x^3 - 2x + 4 = 0$, ſignorum ſubſcriptorum ſeries ſecundum regulam *Newtonianam*, indicaret nullas dari radices impoſſibiles, licet factorem quadraticum habet $x^2 - 2x + 2 = 0$ cujus radices per eandem reg. ſunt impoſſibiles.

## De transmutationibus æquationum.

153. I. *Æquationis cujuſvis radices omnes affirmativas in negativas, & negativas in affirmativas mutare.*

Fiet hoc mutando tantum ſigna terminorum alternorum a ſecundo incipientium. Nam ſi ſubſtituatur $-x$ pro $+x$ in æquatione quacunque dimenſionis *paris*, mutabuntur ſigna terminorum alternorum a ſecundo incipientium, quos nempe occupant impares poteſtates $x$; ſignis reliquorum manentibus. At ſi ſubſtituatur $-x$ pro $x$ in æquatione dimenſionis *imparis*, mutabuntur tantum ſigna terminorum alternorum a primo incipientium, adeoque tranſponendo omnes terminos cum ſignis mutatis, quo terminus altiſſimus novæ æquationis evadat affirmativus, mutabuntur, ut in præcedenti caſu, ſigna terminorum alternorum tantum a ſecundo incipientium.



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Sic æquatio $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, cujus ſi ſigna ſpectes, tres ſunt radices affirmativæ, mutando ſigna ſecundi & quarti termini, fit $x^4 + x^3 - 19x^2 - 49x - 30 = 0$, ejuſque radicum una tantum affirmativa, reliquæ negativæ, ut æquum eſt; adeoque in priore caſu, prodeunt radices $+1$, $+2$, $+3$, & $-5$, in poſteriore, $-1$, $-2$, $-3$, $+5$, mutatis nimirum radicibus affirmativis in negativas, & vice verſâ.

Sic quoque æquatio $x^5 - 4x^4 + 4x^3 - 2x^2 - 5x - 4 = 0$, tres radices affirmativas habens, mutando ſigna ſecundi quarti & ſexti termini, fit $x^5 + 4x^4 + 4x^3 + 2x^2 - 5x + 4 = 0$, cujus tres prodeunt radices negativæ, reliquæ affirmativæ. Ex quinque autem radicibus duæ ſunt impoſſibiles, quæ in priore æquatione inter affirmativas latebant, in poſteriore latent inter negativas; ita ut his deductis reſtet unica tantum radix vere negativa.

154. Poſſumus quoque ſupponere radicem æquationis ex cognita & incognita aliqua quantitate *utcunque componi*, & perinde pro eâ ſubſtituere quod æquipollens eſſe fingitur. Et hoc pacto tranſmutationes aliæ eliciendæ ſunt quæ diverſis uſibus inſerviunt.

155. II. *Radices æquationis data quadam quantitate augere, vel diminuere, vel de data quantitate ſubducere.*

Supponamus radicem æquationis æqualem eſſe ſummæ vel differentiæ cognitæ & incognitæ alicujus quantitatis, deinde, ſubſtituendo pro radice in æquatione datâ id quod æquipollens eſſe fingitur, emerget æquatio nova cujus radices evadent majores vel minores radicibus datæ hac cognitâ quantitate. Sic in æquatione $x^3 - px^2 + qx - r = 0$, ſi radices cognita quantitate $c$ augeri vellem, ponendo $x + c = y$, & perinde ſubſtituendo $y - c$ pro $x$, &c. emerget æquatio,

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$$\left.\begin{array}{r} x^3 - \\ - px^2 \\ + qx \\ - r \end{array}\right\} \left.\begin{array}{rrrr} y^3 - 3cy^2 & + 3c^2y & - c^3 \\ - py^2 & + 2pcy & - pc^2 \\ & + qy & - qc \\ & & - r \end{array}\right\} = 0.$$

Et ſimiliter, ſi radices cognita quantitate $c$ diminui vellem, ſubſtituendo $y + c$ pro $x$, aut ſi radices de cognita quantitate $c$ ſubducere vellem, ſubſtituendo $c - y$ pro $x$, fiet quæſitum.

Atque in genere, radices æquationis ($n$) dimenſionum augere, &c. poſſumus cognita quadam quantitate $c$, ſubſtituendo pro poteſtatibus $x$ ſingulis in terminis æquationis datæ poteſtates correſpondentes binomii $y \overset{\sim}{+} c$, ductas in ſingulorum terminorum coefficientes $p$, $q$, $r$, $s$, &c. reſpective.

156. Cæterum æquationis transformatæ termini abſque involutione facile conflari poſſunt.

Nam ſignis terminorum probè obſervatis, ſubſtituendo tantum $c$ pro $x$ in æquatione data habebitur terminus ultimus. Et multiplicando termini ultimi partem unamquamque per indicem $c$ in iſta parte & dividendo factum per $c$, habebitur coefficiens penultimi. Dein rurſus multiplicando penultimi partem unamquamque per indicem $c$ in iſta parte, jam unitate minorem quam prius, & factum dividendo per $2c$, habebitur coefficiens antepenultimi. Atque ita, ſemper multiplicando termini noviſſimi partem unamquamque per numeros unitate minores quam prius, & factum dividendo per $c$ ductum in numeros unitate majores quam prius, habebitur termini inſequentis coefficiens.

157. Atque ita *Newtonus* methodum generalem inveſtigandi *limites* inter quos conſiſtunt radices æquationum docens, brevitatis ſtudioſus, ex ipſa æquatione data, tan-

quam ex termino ultimo, coefficientes terminorum reliquorum æquationis transformatæ immediate deducit. *Multiplicando utique æquationis terminum unumquemque per numerum dimenſionum ejus, & dividendo factum per radicem æquationis, &c.* Niſi quod facta ſucceſſive prodeuntia dividit per radicem æquationis ſolam, quæ dividi debuerunt per duplam radicem, triplam radicem, &c. reſpective. Et proinde, minuendi gratiâ, terminos reſultantes per maximos diviſores communes, ſiquos ſortiuntur, inſuper dividit.

158. 1. *Hoc pacto efficere poſſumus ut radicum aliquæ vel omnes quæ prius erant negativæ evadant affirmativæ*, augendo utique radices donec augmentum $c$ aliquos vel omnes radices negativas tranſcendat; *vel etiam ut ex affirmativis aliquæ vel omnes evadant negativæ*, minuendo utique radices, donec decrementum, &c. Nam ſi omnis valor $y$ fiat affirmativus, neceſſe eſt ut $x + c = y$, ſit affirmativus, adeoque ut $c$ majus ſit omni radice negativa $x$. Atque ita in cæteris.

Sic in æquatione $x^4 - x^3 - 19x^2 + 49x - 30 = 0$, ſi radices unitate augeri vellem, ſcribendo $y - 1$ pro $x$, æquationis prodeuntis $y^4 - 5y^3 - 10y^2 + 80y - 96 = 0$, radices erunt 2, 3, 4, $-4$, quæ prius erant 1, 2, 3, $-5$, unitate jam factæ majores. Pro $x$ vero ſcribendo $y - 6$, prodiiſſet æquatio $y^4 - 25y^3 + 215y^2 - 695y + 504 = 0$, cujus radices fuiſſent 7, 8, 9, 1, omnes nimirum affirmativæ. Quod ſi pro $x$ ſcripſiſſem $y + 1\frac{1}{2}$ prodiiſſet æquatio $y^4 + 5y^3 - 10y^2 - \frac{5}{4}y + \frac{39}{16} = 0$, cujus radices ita diminutæ evaſiſſent $-\frac{1}{2}$, $+\frac{1}{2}$, $+1\frac{1}{2}$, $-6\frac{1}{2}$, duæ nimirum affirmativæ ac duæ negativæ. Et pro eodem ſcribendo $y + 4$, radices jam numero quaternario diminutæ in æ-

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quatione reſultante $y^4 + 15y^3 + 65y^2 + 105y + 54 = 0$, evaſiſſent $-3$, $-2$, $-1$, $-9$, negativæ omnes.

159. 2. *Et hoc modo augendo vel diminuendo radices ſiquæ impoſſibiles ſunt, hæ aliquando detegentur.*

Sic in æquatione $x^3 - 3a^2x - 3a^3 = 0$, erit ſeries ſignorum $+$, $\pm$, $-$, $-$, ſcilicet in utroque caſu, ſi ſigna ſpectes, una radix affirmativa, reliquæ negativæ. At ſi augeas radices quantitate $a$, ſcribendo $y - a$ pro $x$, in æquatione reſultante $y^3 - 3ay^2 - a^3 = 0$, erit ſeries ſignorum $+$, $-$, $\pm$, $-$, nimirum $+$, $-$, $+$, $-$, vel $+$, $-$, $-$, $-$, adeoque in priore caſu tres radices affirmativæ, in poſteriore una tantum. Sunt ergo duæ impoſſibiles, quæ pro ambiguitate ſua modo inter affirmativas, modo inter negativas latent.

160. 3. *Eadem operatione poſſumus etiam ſecundos terminos æquationum tollere.*

Nam ſi ſecundus terminus æquationis ſit negativus, conſtat per Cor. II. quod radices affirmativæ præpollent negativis, radices igitur minuendæ ſunt donec affirmativæ æquipollent negativis; aut ſi terminus ſecundus ſit affirmativus, radices augendæ ſunt donec negativæ æquipollent affirmativis; hoc vero fiet in utroque caſu, ſi coefficientem ſecundi termini æquationis propoſitæ cum proprio ſigno per numerum dimenſionum æquationis diviſam ſubducamus de quantitate quæ pro novæ æquationis radice ſignificanda aſſumitur, & reſiduum ſubſtituamus pro radice æquationis propoſitæ. Nam

Proponatur æquatio $x^n - px^{n-1} + qx^{n-2}$ &c. $= 0$, cujus ſecundus terminus eſt negativus; fiat $x - c = y$, &

ſubſtituendo $y+c$ pro $x$, proveniet æquatio $y^n \left.\begin{matrix} +nc \\ -p \end{matrix}\right\} y^{n-1}$

$$\left.\begin{matrix} +n\,.\,\overline{\frac{n-1}{2}}\,c^2 \\ -p\,.\,\overline{n-1}\,c \\ +\qquad q \end{matrix}\right\} y^{n-2}$$ &c. $=0$. Fingatur jam coefficientem ſecundi termini hujus nihil eſſe, ſeu $nc-p=0$, adeoque $c=\frac{p}{n}$; & ſubſtituto hoc valore $c$, fiet $nc-p=\frac{np}{n}-p$ $=p-p=0$, adeoque evaneſcet ſecundus terminus novæ æquationis. Et ſimiliter, ſi ſecundus terminus datæ æquationis ſit affirmativus, vice $y-c$ ſubſtituendo $y-\frac{p}{n}$ pro $x$, evaneſcet $-\frac{np}{n}+p$ coefficiens ſecundi termini æquationis novæ. In omni autem caſu valet hæc regula, ſive radices ſint reales ſive impoſſibiles. Ut ſi proponatur æquatio $x^3-4x^2+4x-6=0$, cujus una radix eſt ſurda, reliquæ impoſſibiles, ſubſtituto $y+\frac{4}{3}$ pro $x$, provenit

$$\begin{array}{rrrr} y^3 + 4y^2 & + \frac{16}{3}\,y & + \frac{64}{27} \\ -4y^2 & - \frac{32}{3}\,y & - \frac{64}{9} \\ & + 4y & + \frac{16}{3} \\ & & -6 \\ \hline y^3 \quad * & -\frac{4}{3}\,y & -\frac{146}{27} & =0. \end{array}$$

161. Exterminatio ſecundi termini reſolutioni faciliori æquationum cubicarum & biquadraticarum inſervit. Hoc pacto quoque, æquatio quadratica affecta ſublato termino ſecundo ad normam quadraticæ puræ reduci poteſt. Sic in æquatione $x^2 - px + q = 0$, ſubſtituendo $y + \frac{1}{2}p$ pro $x$, prodit $y^2 - \frac{1}{4}p^2 + q = 0$, adeoque $y^2 = \frac{1}{4}p^2 - q$, & extracta radice $y = \pm \sqrt{\frac{p^2}{4} - q}$ adeoque $x = y + \frac{1}{2}p = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$, ut antea § 29.

162. 4. *Eadem methodo poteſt & tertius æquationis terminus tolli*, quod in conſtructionibus quibuſdem geometricis poſtulatur; inveniendo nempe valorem $c$ qui coefficientem tertii termini æquationis transformatæ ($n$) dimenſionum deſtruet. Fingatur enim coefficientem tertii termini $n \cdot \frac{\overline{n-1}}{2} c^2 - p \cdot \overline{n-1}\, c + q = 0$, adeoque concinnando, $c^2 - \frac{2p}{n} c + \frac{2q}{n \cdot \overline{n-1}} = 0$. Et per reductionem, erit $c = + \frac{p}{n} \pm \sqrt{\frac{p^2}{n^2} - \frac{2q}{n \cdot \overline{n-1}}}$. Ut ſi tollendus ſit tertius terminus æquationis præcedentis $x^3 - 4x^2 + 4x - 6 = 0$ prodit $c = + \frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{8}{6}} = + \frac{4}{3} \pm \frac{2}{3} = + \frac{2}{3}$ vel $+ 2$, quamobrem vice $y + c$ ſcribi debet $y + \frac{2}{3}$ vel $y + 2$ pro $x$, et in utroque caſu tolletur tertius terminus; nam ſi

pro $x$ ſcribatur $y + \frac{2}{3}$ orietur hæc æquatio $y^3 - y^2 * - \frac{137}{27} = 0$, ſin ſcribatur $y + 2$, orietur hæc $y^3 + 2y^2 * - 6 = 0$.

Simili ratione, tolli poſſit 4$^{tus}$ terminus, inveniendo valorem idoneum $c$ per reſolutionem æquationis cubicæ; quintus terminus per reſolutionem biquadraticæ; & ſic deinceps. Verùm hæ tranſmutationes ob calculum difficilem adhiberi non ſolent.

Cæterum obſervandum eſt, quod hac methodo plures terminos intermedios *ſimul* tollere non licet, cum ſingulis terminis tollendis proprius deſideratur valor $c$. Methodum generalem at valde operoſam, omnes terminos intermedios tollendi, tradit *Bezout*, *Mem. de l'Acad.* an. 1772.

163. III. *Radices æquationum per datos numeros multiplicare vel dividere.*

Si radices æquationis $x^n - px^{n-1} + qx^{n-2}$ &c. $+ u = 0$, multiplicandæ forent per numerum $c$, fiat $cx = y$, deinde ſubſtituendo $\frac{y}{c}$ pro $x$, proveniet æquatio $\frac{y^n}{c^n} - \frac{py^{n-1}}{c^{n-1}} + \frac{qy^{n-2}}{c^{n-2}}$ &c. $+ u = 0$. Sive $y^n - pcy^{n-1} + qc^2y^{n-2}$ &c. $+ c^n u = 0$, cujus radices ſunt $c^{plo}$ majores quam ante. Et rurſus ſi dividendæ ſint radices per $c$, fiat $\frac{x}{c} = y$, de-

inde ſubſtituendo $cy$ pro $x$, proveniet $y^n - \frac{py}{c}^{n-1} + \frac{qy}{c^2}^{n-2}$ &c. $+ \frac{u}{c^n} = 0$.

Hinc colligere licet, quod æquationis radices per datum aliquem numerum multiplicare vel dividere poſſumus, multiplicando vel dividendo ſeriem terminorum æquationis incipiendo a primo termino, per ſeriem numerorum continue proportionalium $1, c, c^2, c^3,$ &c. $c^n$, reſpective.

164. 1. *Hoc pacto, fractiones ſemper tollere poſſumus, & tamen efficere ut terminus altiſſimus æquationis reſultantis coefficiente careat.* Sic multiplicando radices æquationis $x^3 + \frac{x^2}{f} + \frac{x}{g} + \frac{u}{h} = 0$, per $c$, prodit $y^3 + \frac{cy^2}{f} + \frac{c^2y}{g} + \frac{c^3u}{h} = 0$, hoc eſt, ſubſtituendo $fgh$ pro $c$, ſi modo $f$, $g$ & $h$, ſint numeri ad invicem primi, $y^3 + ghy^2 + f^2gh^2y + f^3g^3h^2u = 0$. Quod ſi $f$, $g$ & $h$ non ſint ad invicem primi, valorem depreſſiorem $c$ uſurpare licet. Sic in æquatione præcedente $y^3 - \frac{4}{3}y - \frac{146}{27} = 0$, ſubſtituendo $\frac{1}{3}z$ pro $y$, provenit æquatio $\frac{z^3}{27} - \frac{4z}{9} - \frac{146}{27} = 0$, ſive $\frac{z^3 - 12z - 146}{27} = 0$, & rejecto terminorum communi denominatore, $z^3 - 12z - 146 = 0$, cujus radices ſunt triplo majores radicibus datæ æquationis. Atque ita in æquatione $y^4 - \frac{43}{8}y^2 + \frac{75}{8}y - \frac{851}{256} = 0$, ad

tollendas fractiones ſcribe $\frac{1}{4}z$ pro $y$, & orietur $z^4 - 86z^2 + 600z - 851 = 0$.

165. 2. *Et hoc pacto poſſunt radicales* aliquando *tolli*, ubi eadem quantitas radicalis terminos alternos, vel a primo termino vel a ſecundo incipientes, ingreditur. Sic in æquatione $x^3\sqrt{3} - 2x\sqrt{3} + 1 = 0$, ſcribendo $\frac{y}{\sqrt{3}}$ pro $x$, prodit $\frac{y^3\sqrt{3}}{3\sqrt{3}} - \frac{2y\sqrt{3}}{\sqrt{3}} + 1 = 0$, ſive $y^3 - 6y + 3 = 0$. Itemque in æquatione $x^3 - p\sqrt{a}\,x^2 + qx - r\sqrt{a} = 0$ ſubſtituendo $\frac{y}{\sqrt{a}}$ pro $x$, emergit $y^3 - pay^2 + qay - ra^2 = 0$. Sic et in æquatione $x^3 - 2x + \sqrt{3} = 0$, ſcribendo $y\sqrt{3}$ pro $x$, prodit $3y^3\sqrt{3} - 2y\sqrt{3} + \sqrt{3} = 0$, ſeu $3y^3 - 2y + 1 = 0$.

166. IV. *Æquationis radices in earum reciprocas tranſmutare.*

Sic æquatio noviſſima $3y^3 - 2y + 1 = 0$, cujus prima radix eſt $-1$, & reliquæ $\frac{1 \pm \sqrt{-\frac{1}{3}}}{2} = \frac{1 \pm \frac{1}{3}\sqrt{-3}}{2} = \frac{3 \pm \sqrt{-3}}{6}$; ſcribendo $\frac{1}{z}$ pro $y$, evadit $\frac{3}{z^3} - \frac{2}{z^2} + 1 = 0$, ſive concinnando & ordinando, $z^3 - 2z^2 + 3 = 0$, ejuſque prima radix $-1$, & reliquæ $\frac{3 \pm \sqrt{-3}}{2} = \frac{6 \times \overline{3 \pm \sqrt{-3}}}{12} = \frac{6}{3 \mp \sqrt{-3}}$ nempe radicum præcedentium reciprocæ.

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167. 1. *Hoc pacto radix minima in maximam converti poteſt, & maxima in minimam*; quod uſum nonnullum habere poteſt in limitibus radicum exquirendis.

168. 2. *Poteſt etiam æquationis terminus penultimus hac operatione tolli, ſi modo ſecundus prius tollatur*; ut factum vides in exemplo noviſſimo. Aut ſi antepenultimum tolli cupias, id fiet ſi modo tertium prius tollas.

169. V. *Invenire æquationem cujus radices ſint quadrata radicum datæ æquationis.*

Proponatur æquatio cubica $x^3 - Px^2 + Qx - R = 0$, cujus radices ſint $a$, $b$, $c$, maximam deſignante $a$; & mutando ſigna terminorum alternorum orietur æquatio $x^3 + Px^2 + Qx + R = 0$, cujus radices idcirco erunt $-a$, $-b$, $-c$. Deinde his æquationibus in ſe ductis, emerget æquatio 6 dimenſionum $(x^3 + Qx)^2 - (Px^2 + R)^2$ ſeu $x^6 + (2Q - P^2)x^4 + (Q^2 - 2PR)x^2 - R^2 = (x-a)(x+a)(x-b)(x+b)(x-c)(x+c)$ ſeu $(x^2 - a^2)(x^2 - b^2)(x^2 - c^2)$, cujus termini alterni deſunt; evaneſcentibus nimirum ſummâ radicum $a, -a, b, -b$, &c. ſummâ contentorum ſub ſingulis ternis, $-aab$, $+aab$, &c. ſub ſingulis quinis, $+aabbc$, $-aabbc$, &c. ob dimenſionem æquationis reſultantis ſemper parem. Minuendi gratiâ igitur ſubſtitue hinc inde $y$ pro $x^2$, & deprimetur æquatio reſultans ad cubicam $y^3 + (2Q - P^2)y^2 + (Q^2 - 2PR)y - R^2 = (y - a^2)(y - b^2)(y - c^2)$, cujus radices ſunt $a^2$, $b^2$, $c^2$, ut manifeſtum eſt. Et ſic univerſaliter: Sit æquatio $x^n - Px^{n-1} + Qx^{n-2} - Rx^{n-3}$ + &c. $= 0$, eritque æquatio reſultans $y^n + (2Q - P^2)y^{n-1}$ $+ (2S - 2PR + Q^2)y^{n-2}$ + &c. $= 0$.

170. Porro, hanc operationem inſtituendo in æquationem *quadratorum*, tanquam datam, emerget æquatio cujus radices erunt *biquadrata* radicum æquationis datæ; eandemque iterando in æquationem proxime inventam, tanquam datam, æquationes *per ſaltum* invenire licet cujus radices ſint *poteſtates* $8^{væ}$, $16^{mæ}$, $32^{mæ}$, $64^{mæ}$, &c. radicum datæ æquationis, abſque tædio omnes poteſtates intermedias radicum inveniendi, per Formulas, § 79.

Sic æquatio propoſita, ſublato termino ſecundo, erit $x^3 + Qx - R = 0$, unde, evaneſcentibus terminis æquationum reſultantium quos intrant dimenſiones P, erunt *

$$y^3 + 2Qy^2 + Q^2y - R^2 = 0, \quad \textit{æquat. quadrat.}$$

$$y^3 - 2Q^2y^2 + (Q^4 + 4QR^2)y - R^4 = 0, \quad \textit{æquat. 4 poteſt.}$$

$$\left.\begin{aligned} &y^3 - (2Q^4 - 8QR^2)y^2 + \\ &\quad + (Q^8 + 8Q^5R^2 + 12Q^2R^4)y - R^8 \end{aligned}\right\} = 0, \textit{ æq. 8 poteſt.}$$

$$y^3 - (2Q^8 - 48Q^5R^2 + 40Q^2R^4)y^2 + \text{\&c.} = 0, \textit{ æq. 16 pot.}$$

$$y^3 - \left.\begin{aligned} &(2Q^{16} - 224Q^{13}R^2 + 2288Q^{10}R^4 \\ &-4224Q^7R^6 + 1320Q^4R^8 - 32QR^{10} \end{aligned}\right\} \begin{aligned} &y^2 + \text{\&c.} = 0, \\ &\textit{æq. 32 poteſt.} \end{aligned}$$

$$y^3 - \left.\begin{aligned} &(2Q^{32} - 960Q^{29}R^2 + 58464Q^{26}R^4 \\ &- 1048320Q^{23}R^6 + 7104240Q^{20}R^8 \\ &- 19997120Q^{17}R^{10} + 23772800Q^{14}R^{12} \\ &- 11410944Q^{11}R^{14} + 1961256Q^8R^{16} \\ &- 93632Q^5R^{18} + 672Q^2R^{20} \end{aligned}\right\} \begin{aligned} &y^2 + \text{\&c.} = 0, \\ &\textit{æq. 64 pot.} \end{aligned}$$

* Vide *Bertrand*, *Elem. Mathematiques*; *Geneva*, 1778, vol. II, p. 501.

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Unde mutando ſigna terminorum ſecundorum æquationum reſultantium habebitur ſumma poteſtatum 2, 4, 8, 16, 32, 64, &c. radicum æquationis datæ.

171. *Et hoc pacto, radices impoſſibiles, ſiquæ ſunt, utplurimum detegere poſſumus.*

Nam ſi æquationis datæ nullæ radices impoſſibiles ſint, ſingulorum radicum quadrata erunt affirmativa, adeoque termini æquationis quadratorum, vicibus alternis affirmativi & negativi. Ubi igitur in æquatione quadratorum occurrit ſignorum ſimilium ſucceſſio, indicio eſt æquationem datam radices impoſſibiles ſortiri. Et hoc pacto ſemper detegetur impoſſibilitas radicum æquationis cujuſcunque ubi P deleto, Q ſit affirmativum. Sic habitâ æquatione $x^3 + 6x - 20 = 0$, prodit æquatio quadratorum $y^3 + 12y^2 + 36y - 400 = 0$, quæ duas habet ſignorum ſimilium ſucceſſiones. Ergo, &c. Quòd ſi Q negativum ſit, licet radices ſint impoſſibiles, nulla deprehendi poteſt ſignorum ſucceſſio in æquatione quadratorum; verùm operationem continuando, in cubicis ſaltem, impoſſibilitatem radicum ſemper deteget æquatio biquadratorum, ubi $R^2$ ſit majus quam $\frac{4}{27}Q^3$, evadente tertio termino hujus $(Q^4 + 4QR^2)\ y$ negativo. Et ſimiliter in æquationibus altioribus caſus determinare licet, ubi radicibus impoſſibilibus detegendis impar erit hæc methodus. Sic habitâ æquatione $x^3 - 2x + 4 = 0$, prodit æquatio quadratorum $y^3 - 4y^2 + 4y - 16 = 0$, quæ nullam habet ſignorum ſimilium ſucceſſionem; at in æquatione biquadratorum $y^3 - 8y^2 - 112y - 256 = 0$, duæ prodeunt ſucceſſiones. Ergo, &c.

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172. *Numerus autem ſucceſſionum ſimilium ſignorum ſemper æqualis vel minor erit numero radicum impoſſibilium.*

Sint enim radices impoſſibiles $a + b\sqrt{-1}$ & $a - b\sqrt{-1}$, harum quadrata erunt $a^2 - b^2 + 2ab\sqrt{-1}$ & $a^2 - b^2 - 2ab\sqrt{-1}$ reſpective, ſive, (poſito $a^2 - b^2 = \text{A}$, & $2ab = \text{B}$) $\text{A} + \text{B}\sqrt{-1}$ & $\text{A} - \text{B}\sqrt{-1}$ *; ubi A erit affirmativum vel negativum, prout $a^2$ ſit majus minuſve quam $b^2$. At multiplicando factores $x \pm \text{A} + \text{B}\sqrt{-1}$ & $x \pm \text{A} - \text{B}\sqrt{-1}$ in ſe generabitur æquatio quadratica $x^2 \pm 2\text{A}x \begin{matrix} + \text{A}^2 \\ + \text{B}^2 \end{matrix} = 0$,

* Radices binas impoſſibiles æquationis algebraicæ ad formam $\text{A} \pm \text{B}\sqrt{-1}$ reduci poſſe, primus, ex calculo integrali demonſtravit *D'Alembert*, *Mem. Acad. Berlin*, 1746. Hunc ſecutus eſt *Euler*, *Mem.* 749; & *Foncenex*, *Mem. Turin*, 1760; cujus demonſtrationem ex principiis trigonometricis petitam impugnat *La Grange*, *Mem. Berlin*, 1772, aliamque proponit; contra vero tuetur *Bertrand*, *Elemen. des Mathematiques*, vol. 2, p. 489. Hujus reductionis adeo decantatæ veritatem, ex principiis algebraicis ſic colligere licet.

Æquationis quadraticæ $x^2 - px + q = 0$, radices impoſſibiles ſunt $\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$. Abbreviandi gratiâ, ſit $\frac{1}{4}p^2 - q = -r$, eritque $\sqrt{\frac{1}{4}p^2 - q} = \sqrt{-r} = \sqrt{r} \cdot \sqrt{-1}$, adeoque poſito $\frac{1}{2}p = a$ & $\sqrt{r} = b$, radices quadraticæ impoſſibiles formam induent $a \pm b\sqrt{-1}$. Et quoniam æquationis cujuſcunque algebraicæ radices impoſſibiles conflari neceſſe eſt per additionem, ſubductionem, multiplicationem, diviſionem, involutionem aut evolutionem quantitatum impoſſibilium ſive ſimplicium ſive mixtarum; deſignantibus A & B quantitatum per haſce operationes reſultantium partes reales cum propriis ſignis, erit,

I. $\begin{matrix} a + b\sqrt{-1} \\ \pm g + b\sqrt{-1} \end{matrix} = (a \pm g) + (b \pm b)\sqrt{-1} = \text{A} + \text{B}\sqrt{-1}.$

II. $\begin{matrix} a + b\sqrt{-1} \\ \times g + b\sqrt{-1} \end{matrix} = (ag - bb) + (bg + ab)\sqrt{-1} = \text{A} + \text{B}\sqrt{-1}.$

III. $\frac{a + b\sqrt{-1}}{g + b\sqrt{-1}} = \frac{(a + b\sqrt{-1})\,(g - b\sqrt{-1})}{(g + b\sqrt{-1})\,(g - b\sqrt{-1})} = \frac{(ag + bb) + (bg - ab)\sqrt{-1}}{g^2 - b^2}$
$= \text{A} + \text{B}\sqrt{-1}.$

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in qua ſucceſſiones ſignorum ſimilium ſunt duæ vel nullæ; ergo, &c. Et ſimiliter in æquationibus altioribus demonſtrare licet.

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IV. $(a+b\sqrt{-1})^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n} a^{\frac{m}{n}-1} (b\sqrt{-1}) + \frac{\frac{m}{n} \cdot \frac{m}{n}-1}{2} a^{\frac{m}{n}-2} (b\sqrt{-1})^2 + \frac{\frac{m}{n} \cdot \frac{m}{n}-1 \cdot \frac{m}{n}-2}{2 \cdot 3} a^{\frac{m}{n}-3} (b\sqrt{-1})^3 + \frac{\frac{m}{n} \cdot \frac{m}{n}-1 \cdot \frac{m}{n}-2 \cdot \frac{m}{n}-3}{2 \cdot 3 \cdot 4} a^{\frac{m}{n}-4} (b\sqrt{-1})^4$ &c.

Hujus ſeriei terminos impares occupant poteſtates pares quantitatis $b\sqrt{-1}$, & vice verſa. Sed poteſtates pares evadunt $-b^2$, $+b^4$, $-b^6$, &c. omnes nimirum reales, impares vero evadunt $+b\sqrt{-1}$, $-b^3\sqrt{-1}$, $+b^5\sqrt{-1}$, &c. ſeu $(+b-b^3+b^5-\&c.)\sqrt{-1}$. Adeoque aggregatum terminorum ſeriei imparium $a^{\frac{m}{n}} - \frac{\frac{m}{n} \cdot \frac{m}{n}-1}{2} a^{\frac{m}{n}-2} b^2 + \&c.$ deſignare poteſt A; parium vero $\left(\frac{m}{n} a^{\frac{m}{n}-1} b - \frac{\frac{m}{n} \cdot \frac{m}{n}-1 \cdot \frac{m}{n}-2}{2 \cdot 3} a^{\frac{m}{n}-3} b^3 + \&c.\right)\sqrt{-1}$, deſignare poteſt $B\sqrt{-1}$.

Radix igitur ex his operationibus reſultans ſemper prodibit formæ $A+B\sqrt{-1}$, cui germana exſurget formæ $A-B\sqrt{-1}$, per Cor. VII. Ergo, &c.

Eadem quoque reductio valet in impoſſibilibus *exponentialibus*, ſeu quarum indices ſunt irrationales vel impoſſibiles, velut $(a \pm b\sqrt{-1})^{g+h\sqrt{-1}}$. Hunc vero caſum prætermittimus, quippe qui in Elementis Algebræ parum ſit utilis, ejuſque demonſtratio non niſi ex principiis trigonometricis & logiſticis petenda videtur. Si quis tamen deſiderat, conſulat *Auct. cit.* aut *Fenn's Elements of Specious Arithm. Dublin*, 1772, *p.* 222.

173. VI. *Invenire æquationem cujus radices ſint quadrata differentiarum radicum datæ æquationis.*

Proponatur æquatio $x^3 - Px^2 + Qx - R = 0$, cujus radices ſint $a, b, c$; erunt differentiæ radicum $a - b$, $b - a$, $a - c$, $c - a$, $b - c$, $c - b$; ſeu abbreviandi gratiâ, $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$. Adeoque erit $(y - \alpha)(y + \alpha)(y - \beta)(y + \beta)(y - \gamma)(y + \gamma)$ ſeu $(y^2 - \alpha^2)(y^2 - \beta^2)(y^2 - \gamma^2) = y^6 - P'y^4 + Q'y^2 - R' = 0$, evaneſcentibus nimirum terminis alternis. Subſtitue iterum $z$ pro $y^2$, & emerget $(z - \alpha^2)(z - \beta^2)(z - \gamma^2) = z^3 - P'z^2 + Q'z - R' = 0$, cujus radices ſunt $\alpha^2, \beta^2, \gamma^2$, quadrata nempe differentiarum radicum datæ æquationis, ut manifeſtum eſt. Et ſic univerſaliter: æquationis $x^n - Px^{n-1} + Qx^{n-2}$ — &c. $= 0$, erit numerus differentiarum radicum $= n \times \overline{n-1}$, & proinde æquatio quadratorum differentiarum erit $n \cdot \frac{n-1}{2}$ dimenſionum.

Jam quo eruantur coefficientes, $P'$, $Q'$, $R'$, erit

Summa radicum $(a - b)^2 + (a - c)^2 + (b - c)^2 =$

$$= 2(a^2 + b^2 + c^2) - 2(ab + ac + bc) =$$

Adeoque $P'$ . . $= 2(P^2 - 2Q) - 2Q = 2P^2 - 6Q$.

Summa rect. ſub binis rad. $(a - b)^2 (a - c)^2 + (a - b)^2 (b - c)^2 + (a - c)^2 (b - c)^2$

$$= \begin{cases} a^4 - 2a^3(b + c) + 3a^2b^2 \\ b^4 - 2b^3(a + c) + 3a^2c^2 \\ c^4 - 2c^3(a + b) + 3b^2c^2 \end{cases}$$

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1. $a^4 + b^4 + c^4 = P^4 - 4P^2Q + 4PR + 2Q^2.$

2. $-2\left(a^3 \times \overline{b+c} + b^3 \times \overline{a+c} + c^3 \times \overline{a+b}\right) =$

$= -2\left(a^3 \times \overline{P-a} + b^3 \times \overline{P-b} + c^3 \times \overline{P-c}\right) =$

$= -2\left(\overline{a^3+b^3+c^3} \times P - \overline{a^4+b^4+c^4}\right) = -2P^2Q + 2PR + 4Q^2.$

3. $+3\left(a^2b^2 + a^2c^2 + b^2c^2\right) = 3\left(\overline{ab+ac+bc}^2 - 2abc \times \overline{a+b+c}\right) =$

$= 3(Q^2 - 2PR) = 3Q^2 - 6PR.$

$$\text{Adeoque } Q' = \left\{\begin{array}{rrrr} P^4 & -\,4P^2Q & +\,4PR & +\,2Q^2 \\ & -\,2P^2Q & +\,2PR & +\,4Q^2 \\ & & -\,6PR & +\,3Q^2 \\ \hline P^4 & -\,6P^2Q & * & +\,9Q^2. \end{array}\right.$$

Summa content. ſub ternis rad. $(a-b)^2\ (a-c)^2\ (b-c)^2.$

$$= \left\{\begin{array}{lllll} a^4b^2 & -\,2a^4bc & +\,2a^3b^2c & -\,2a^3b^3 & -\,6a^2b^2c^2. \\ a^4c^2 & -\,2b^4ac & +\,2a^3bc^2 & -\,2a^3c^3 & \\ b^4a^2 & -\,2c^4ab & +\,2b^3a^2c & -\,2b^3c^3 & \\ b^4c^2 & & +\,2b^3ac^2 & & \\ c^4a^2 & & +\,2c^3a^2b & & \\ c^4b^2 & & +\,2c^3ab^2 & & \end{array}\right.$$

1. $a^4 \times \overline{b^2+c^2} + b^4 \times \overline{a^2+c^2} + c^4 \times \overline{a^2+b^2} =$

$= a^4 \times \overline{P^2 - 2Q - a^2} + b^4 \times \overline{P^2 - 2Q - b^2} + c^4 \times \overline{P^2 - 2Q - c^2} =$

$= (a^4 + b^4 + c^4)\,P^2 - (a^4 + b^4 + c^4)\,2Q - (a^6 + b^6 + c^6) =$

$= -2P^3R + P^2Q^2 + 4PQR - 2Q^3 - 3R^2.$



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2. $-2(a^3+b^3+c^3)abc = -2(P^3 - 3PQ + 3R)R =$
$= -2P^3R + 6PQR - 6R^2.$

3. $2abc(a^2 \times \overline{b+c} + b^2 \times \overline{a+c} + c^2 \times \overline{b+c}) =$
$= 2R(\overline{a^2+b^2+c^2} \times P - \overline{a^3+b^3+c^3}) = 2PQR - 6R^2.$

4. $-2(a^3b^3 + a^3c^3 + b^3c^3) =$
$= -2(\overline{ab+ac+bc}^3 - 3(a^2 \times \overline{b+c} + b^2 \times \overline{a+c} + c^2 \times \overline{a+b}) -$
$- 6a^2b^2c^2) = 6PQR - 2Q^3 - 6R^2.$

5. $-6a^2b^2c^2 = -6R^2.$

Adeoque harum ſumma ſeu $R' = -4P^3R + P^2Q^2 + 18PQR - 4Q^3 - 27R^2$. Unde prodit æquatio quadr. different. $z^3 - (2P^2 - 6Q)z^2 + (P^4 - 6P^2Q + 9Q^2)z + 4P^3R - P^2Q^2 - 18PQR + 4Q^3 + 27R^2 = 0.$

174. Tollendo autem terminum ſecundum datæ æquationis, formam induet ſimpliciorem æquatio quadratorum differentiarum, nempe

$$z^3 + 6Qz^2 + 9Q^2z + 4Q^3 + 27R^2 = 0.$$

175. Proponatur æquatio biquadratica $x^4 * + Qx^2 - Rx + S = 0$, eritque æquatio reſultans $z^6 - P'z^5 + Q'z^4 - R'z^3 + S'z^2 - T'z + V' = 0.$

Jam quo eruantur hujus coefficientes, erit

$P' = (a-b)^2 + (a-c)^2 + (a-d)^2 + (b-c)^2 + (b-d)^2 + (c-d)^2$
$= 3(a^2+b^2+c^2+d^2) - 2(ab+ac+ad+bc+bd+cd)$
$= 3(* - 2Q) - 2Q = -8Q.$

$Q' = (a-b)^2 (a-c)^2 + (a-b)^2 (a-d)^2 + \&c.$
$+ (a-c)^2 (b-c)^2 + \&c.\ \&c.$

$$= \begin{cases} 3a^4 - 4a^3 \times \overline{b+c+d} + 8a^2b^2 - 2a^2 \times \overline{bc+bd+cd} + 12abcd, \\ 3b^4 - 4b^3 \times \overline{a+c+d} + 8a^2c^2 - 2b^2 \times \overline{ac+ad+cd} \\ 3c^4 - 4c^3 \times \overline{a+b+d} + 8a^2d^2 - 2c^2 \times \overline{ab+ad+bd} \\ 3d^4 - 4d^3 \times \overline{a+b+c} + 8b^2c^2 - 2d^2 \times \overline{ab+ac+bc} \\ \qquad + 8b^2d^2 \\ \qquad + 8c^2d^2 \end{cases}$$

1. $3(a^4 + b^4 + c^4 + d^4) = 3(2Q^2 - 4S) = 6Q^2 - 12S.$
2. $-4(a^3 + b^3 + c^3 + d^3) P - (a^4 + b^4 + c^4 + d^4)$
   $= 4(2Q^2 - 4S) = 8Q^2 - 16S.$
3. $+8(a^2b^2 + a^2c^2 + \&c.) = 8(Q^2 + 2S) = 8Q^2 + 16S.$
4. $-2a(abc + abd + acd) - 2b(abc + \&c.) - \&c.\ \&c.$
   $= -2(a+b+c+d)R + 8abcd = +8S.$
5. $+12abcd = 12S.$ Adeoque $Q' = 22Q^2 + 8S.$

Et cæteris coefficientibus $R'$, $S'$, $T'$, $V'$ ſimiliter inveſtigatis, prodit æquatio quadratorum differentiarum,

$$z^6 + 8Qz^5 + (22Q^2 + 8S)z^4 + (28Q^3 + 26R^2 + 16QS)z^3$$
$$+ (17Q^4 + 48R^2Q - 112S^2 + 24Q^2S)z^2 + (4Q^5 + 18Q^2R^2$$
$$+ 32Q^3S - 192QS^2 + 216R^2S)z + 256S^3 - 128Q^2S^2$$
$$+ 144R^2QS + 16Q^4S - 4R^2Q^3 - 27R^4 = 0.$$

176. Si proponatur æquatio quinque dimenſionum $x^5 + qx^3 - rx^2 + sx - t = 0$, erit æquatio quadratorum differentiarum radicum,

$$z^{10} + 10qz^9 + (39q^2 + 10s)\,z^8 + (80q^3 + 50qs + 25r^2)\,z^7$$
$$+ (95q^4 + 124q^2s - 95s^2 + 92qr^2 + 200rt)\,z^6 + (66q^5$$
$$- 360qs^2 + 196q^3s + 118q^2r^2 + 260r^2s + 625t^2 + 400qrt)\,z^5$$
$$+ (25q^6 + 40s^3 - 53r^4 + 52q^3r^2 - 522q^2s^2 + 194q^4s +$$
$$708qr^2s + 240q^2rt + 1750qt^2 - 950srt)\,z^4 + (4q^7 + 106q^5s$$
$$- 80qs^3 - 308q^3s^2 - 102qr^4 - 7q^4r^2 + 570r^2s^2 + 612q^2r^2s$$
$$+ 700r^3t - 3750t^2s + 2500t^2q^2 + 80q^3rt - 2150qrst)\,z^3 +$$
$$(400s^4 - 360q^2s^3 - 15q^4s^2 + 24q^6s - 8q^5r^2 - 45q^2r^4 -$$
$$270r^4s + 140q^3r^2s + 960qr^2s^2 + 1875r^2t^2 + 1000rts^2 -$$
$$5000qst^2 + 1750q^3t^2 + 40q^4rt + 600qtr^3 - 1650q^2rst)\,z^2 +$$
$$(36q^5s^2 - 224q^3s^3 + 320qs^4 + 4q^3r^4 + 27r^6 - 40r^2s^3 +$$
$$434q^2r^2s^2 - 24q^4r^2s - 198qr^4s + 5000s^2t^2 - 450r^3st -$$
$$6250rt^3 + 675q^4t^2 - 3750q^2st^2 + 3000qr^2t^2 + 60q^2r^3t +$$
$$200qrts^2 - 330q^3rst)\,z + 3125t^4 - 3750qrt^3 + \overline{2000qs^2 + 2250r^2s}$$
$$\overline{- 900sq^3 + 825q^2r^2 + 108q^5} \times t^2 - \overline{1600rs^3 - 560q^2rs^2 - 16q^3r^3 +}$$
$$\overline{630qsr^3 + 72q^4rs - 108r^5} \times t + \overline{256s^3 - 128q^2s^2 + 144qr^2s + 16q^4s}$$
$$\overline{- 4q^3r^2 - 27r^4} \times s^2 = 0.$$

177. Et ſic univerſaliter: propoſita æquatione $x^n - Px^{n-1} + Qx^{n-2} - \&c. = 0$, æquationis reſultantis $z^{n \cdot \overline{\frac{n-1}{2}}} - P'z^{n \cdot \overline{\frac{n-1}{2}} - 1} + Q'z^{n \cdot \overline{\frac{n-1}{2}} - 2} - \&c. = 0$, erunt coefficientes

$$P' = \left\{ \begin{array}{l} \overline{n-1}\,(a^2 + b^2 + c^2 + \&c.) = (n-1)\,\overline{P^2 - 2Q} \\ -\,2\,(ab + ac + ad + \&c.) = \qquad\qquad -\,2Q \end{array} \right\} = \overline{n-1}\,P^2 - 2nQ.$$

$$Q' = \left\{ \begin{array}{l} \overline{n-1}\,.\,\overline{\tfrac{n-2}{2}}\,(a^4 + b^4 + c^4\ \&c.) = \overline{n-1}\,.\,\overline{\tfrac{n-2}{2}}\,(P^4 - 4P^2Q + 4PR + 2Q^2 - 4S) \\ -\,2\,.\,\overline{n-2}\,(a^3 \times \overline{b+c+d\ \&c.} + b^3 \times \overline{a+c+d\ \&c.} + \&c.) = -\,2\,.\,\overline{n-2}\,(P^2Q - PR - 2Q^2 + 4S) \\ -\,2\,.\,\overline{n-3}\,(a^2 \times \overline{bc+bd+cd}\ \&c. + b^2 \times \overline{ac+ad+\&c.} + \&c.) = -\,2\,.\,\overline{n-3}\,(PR - 4S) \\ +\,n\,.\,\overline{n-2}\,(a^2b^2 + a^2c^2 + a^2d^2 + \&c.) = n\,.\,\overline{n-2}\,(-\,2PR + Q^2 + 2S) \\ +\,12\,(abcd + abce + \&c.) = 12S. \end{array} \right.$$

$$= \overline{n-1}\,.\,\overline{\tfrac{n-2}{2}}\,P^4 - 2n\,.\,\overline{n-2}\,P^2Q - 2\,.\,\overline{n-3}\,PR + \overline{2n+3}\,.\,\overline{n-2}\,Q^2 + 2nS.$$

Et ſic deinceps.

178. 1. *Hac operatione radices impoſſibiles, ſiquæ ſunt, earumque numerum ſemper detegere poſſumus.*

Nam ſi omnes datæ æquationis radices ſint reales, neceſſe eſt ut quadrata differentiarum radicum ſint omnia affirmativa, adeoque æquationis reſultantis termini vicibus alternis affirmativi & negativi. Quòd ſi data æquatio radices impoſſibiles ſortiatur, cum differentia duarum correſpondentium radicum impoſſibilium $a + b\sqrt{-1}$ & $a - b\sqrt{-1}$, vel $+ b\sqrt{-1}$ & $- b\sqrt{-1}$, ſit $2b\sqrt{-1}$, erit hujus quadratum $-4b^2$, reale nempe & negativum; unde æquationi reſultanti competit una ſaltem radix realis & negativa, et proinde una ſaltem ſucceſſio ſignorum ſimilium. Atque hinc numerus radicum impoſſibilium in data æquatione *duplum* radicum negativarum, ſeu ſucceſſionum ſignorum ſimilium in æquatione reſultante nunquam tranſcendere poteſt.

179. Ut ſi proponatur æquatio cubica $x^3 + qx - r = 0$, ubi $q$ affirmativum eſt, vel ubi $q$ negativum eſt, & $4q^3$ ſimul minus quam $27r^2$, emerget in æquatione reſultante § 174 ſucceſſio ſimilium ſignorum, adeoque æquationis datæ duæ erunt radices impoſſibiles, tertia autem realis & affirmativa erit ſi $r$ ſit negativum, et vice verſâ, per § 146. Sic habitâ æquatione $x^3 - 2x + 4 = 0$, erit æquatio quadratorum differentiarum $z^3 - 12z^2 + 36z + 300 = 0$, adeoque ob ſucceſſionem ſimilium ſignorum, datæ æquationis duæ erunt radices impoſſibiles, tertie autem negativa. In cubicis igitur ſemper definietur numerus radicum impoſſibilium, utpote qui binarium tranſcendere nequit.

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180. Si proponatur æquatio biquadratica $x^4 + qx^2 - rx + s = 0$, ubi in æquatione reſultante prodit ſucceſſio ſignorum ſimilium, conſtat æquationem datam radices impoſſibiles ſortiri. Cæterum *numerum* quoque radicum impoſſibilium, terminis rite collatis, eruere licet. Fingatur enim omnes datæ æquationis radices eſſe impoſſibiles, eritque æquationis reſultantis ultimus terminus $256s^3 - 128q^2s^2 +$ &c. affirmativus, cum ex ſex radicibus ejus, binæ quæque correſpondentes radices in ſe ductæ factum referent affirmativum, formæ $B^2$ vel $A^2 + B^2$, adeoque factum ſub omnibus erit affirmativum. Et proinde ſi $256s^3 - 128q^2s^2 +$ &c. emergat negativus, æquationis datæ quatuor radices impoſſibiles eſſe nequeunt.

Hinc æquationis biquadraticæ radices impoſſibiles, ſiquæ hac operatione detegantur, erunt vel quatuor, ubi æquationis reſultantis terminus ultimus eſt affirmativus aut nihil; vel duæ tantum, ubi eſt negativus: & duæ quæ ſuperſunt radices reales, contrariis afficientur ſignis, ubi ultimus datæ æquationis terminus eſt negativus, per § 147.

181. Similiter, æquationis quintæ dimenſionis, quæ neceſſario unam habet radicem realem, erunt radices impoſſibiles quatuor, ubi æquationis reſultantis terminus ultimus eſt affirmativus aut nihil; vel duæ tantum, ubi eſt negativus. Atque iiſdem veſtigiis inſiſtendo, numerum radicum impoſſibilium æquationis cujuſcunque eruere licet.

182. Operoſa certe evadit hæc methodus eruendi radices impoſſibiles, præſertim in æquationibus dimenſionum ſuperiorum. At ubi faciliore uti non licet, difficilem faſtidire haud æquum eſt. Doctrinam radicum impoſſibilium ſiquis penitius excutere velit, auctorem ſupra laudatum,



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at paulo quidem reconditiorem, *Waring*, conſulat, quem in hac re, potiſſimum, ut interpretes ſecuti ſumus. *Med. Algeb. Prob.* 5 & 11. Conſulat quoque *La Grange*, *Acad. Berlin*, *an.* 1767, 1768, & 1772.

183. 2. *Hac operatione quoque, ubi duæ æquationis radices ſunt æquales, detegere poſſumus.*

Nam ubi ultimus æquationis reſultantis terminus evaneſcit, neceſſe eſt ut una ex radicibus ejus evaneſcat, ſive ut quadratum differentiæ duarum radicum datæ æquationis evaneſcat, adeoque ut hæ duæ radices ſint inter ſe æquales. Defectus igitur termini ultimi æquationis reſultantis, indicio eſt æquationem datam duas radices inter ſe æquales ſortiri.

184. Facilius autem ope Tranſmutationis II, caſus detegere licet, ubi duæ, tres, pluresve datæ æquationis radices, ſint inter ſe æquales. Nam ubi ultimus transformatæ terminus evaneſcit, erit una radix ſeu valor $y = 0$, adeoque datæ æquationis una radix ſeu valor $x = \pm c$, ob $x \mp c = y = 0$. Ubi duo poſtremi transformatæ termini evaneſcunt, erunt duæ radices ſeu valores $y = 0$, adeoque datæ æquationis duæ radices $x = \pm c$, adeoque æquales inter ſe. Similiter, ubi tres poſtremi termini deſunt, erunt datæ æquationis tres radices æquales inter ſe. Et ſic deinceps.

185. Hinc ſi datæ æquationis duæ ſint radices inter ſe æquales, transformatæ coefficiens penultima, quæ in hoc caſu fit nihil, ſubſtituendo $x$ pro $c$, æquationem conſtituet dimenſionis depreſſioris cui radix erit $c$, una nempe radicum æqualium datæ æquationis. Si datæ æquationis tres radices ſint inter ſe æquales, coefficiens transformatæ ante-

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penultima æquationem præbebit dimenſionis adhuc depreſſioris, cui radix erit una ex tribus radicibus æqualibus datæ æquationis. Atque ſic deinceps.

His traditis de radicum natura, atque præcipuis affectionibus & tranſmutationibus æquationum, pergamus jam reſolutionem æquationum radices rationales habentium docere.

## De resolutione æquationum radices rationales habentium.

186. Quoniam æquationis cujuſcunque $x^n \cdot Px^{n-1} \cdot Qx^{n-2} \cdot$ &c. $U = 0$, terminos nec fractos nec ſurdos habentis omnes radices non ſurdæ ſunt aliqui ex diviſoribus *integris* termini ultimi; per § 144, Cor. VI. Subſtituendi ſunt in æquatione pro $x$, omnes diviſores ultimi termini tam ſimplices quam compoſiti ſigillatim, et ſi horum aliquis omnes æquationis terminos ſe mutuo deſtruere efficiet, radix æquationis erit. Quòd ſi diviſorum nullus hanc conditionem impleverit, patet nullam eſſe æquationis radicem niſi quæ ſit aut irrationalis aut impoſſibilis.

Invenientur autem omnes diviſores ultimi termini, *ſi quantitas ſimplex eſt, eam dividendo per minimum ejus diviſorem, & quotum per minimum diviſorem ejus, donec quotus reſtet indiviſibilis. Hi diviſores una cum quoto indiviſibili conſtituent omnes quantitatis diviſores primos ſeu ſimplices; et rectangula aut contenta ſub ſingulis binis, ternis, quaternis, &c. horum diviſorum, omnes ejus diviſores compoſitos.*



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Sic in æquatione $x^3 - 4x^2 - 7x + 10 = 0$, diviſores termini ultimi prodeunt $1, -1, 2, -2, 5, -5, 10, -10$; ſcriptis autem ſucceſſive

$$\left.\begin{array}{r} 1 \\ -1 \\ 2 \\ -2 \\ 5 \end{array}\right\} \text{pro } x, \text{ prodit} \left\{\begin{array}{rl} 1 - 4 - 7 + 10 = & 0, \textit{adeoq.} +1 \textit{ erit radix} \\ -1 - 4 + 7 + 10 = & 12, \\ 8 - 16 - 14 + 10 = & -12, \\ -8 - 16 + 14 + 10 = & 0, \textit{adeoq.} -2 \textit{ erit radix} \\ 125 - 100 - 35 + 10 = & 0, \textit{adeoq.} +5 \textit{ erit radix} \end{array}\right.$$

&c. &c.

187. Inventis hac methodo omnibus radicibus æquationis *præter unam*, hæc facilius erui poteſt, dividendo terminum ultimum ſub ſigno mutato, per contentum radicum jam inventarum. Sic in hac æquatione, dividendo $-10$ per $+1 \times -2 = -2$, quotus $+5$ erit tertia radix.

Inventis omnibus radicibus *præter duas*, hæ erui poſſunt ſive ſint reales ſive impoſſibiles, deprimendo æquationem datam ad quadraticam, ope factorum ſimplicium vel compoſitorum ex radicibus jam inventis reſultantium, & hujus reductio dabit duas quæ ſuperſunt radices. Sic inventâ $+1$ radice æquationis cubicæ $x^3 - 4x^2 - 7x + 10 = 0$, dividendo per $x - 1 = 0$ reſultabit quadratica $x^2 - 3x - 10 = 0$, cujus radices ſunt $+5$ vel $-2$. Inventis item $+2$ & $+4$ radicibus biquadraticæ $x^4 - 4x^3 - 8x + 32 = 0$, dividendo per $x - 2 = 0$, et rurſus quotum per $x - 4 = 0$; vel ſimul dividendo per $(x - 2)(x - 4) = 0$, emerget æquatio quadratica $x^2 + 2x + 4 = 0$, cujus radices ſunt $-1 + \sqrt{-3}$ & $-1 - \sqrt{-3}$. Habentur itaque quatuor radices quæſitæ.

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188. *Idem brevius abſolvere licet tollendo ſecundum terminum.*

Sic in æquatione noviſſimâ ſcribendo $y + 1$ pro $x$, prodit $y^4 - 6y^2 - 16y + 21 = 0$, & tentando diviſores termini ultimi 21, duæ prodeunt radices rationales $+ 1$ & $+ 3$, quarum ſumma eſt 4, adeoque ob defectum ſecundi termini, ſumma reliquarum erit $- 4$. Deſignet $v$ unam ex his, alteram deſignabit $- 4 - v$, eritque harum factum $- 4v - v^2 = \frac{21}{3} = 7$, adeoque $v^2 + 4v = - 7$, & $v = - 2 + \sqrt{-3}$ & altera $- 4 - v = - 2 - \sqrt{-3}$. Prodeunt igitur quatuor radices æquationis novæ $+ 1$, $+ 3$, $- 2 + \sqrt{-3}$, $- 2 - \sqrt{-3}$, quæ unitate auctæ dabunt radices æquationis datæ.

189. Cæterum ubi diviſores ultimi termini ſunt permulti, ut in æquatione $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120 = 0$, cujus ultimi termini diviſores 32 ſunt, nempe 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120, $- 1$, $- 2$, &c. hoſce omnes tentare tædio foret. Laboris igitur minuendi gratiâ, primus excogitavit *De Beaune* methodum inveniendi limites inter quos conſiſtunt radices, quam poſtea perfecit *Newtonus*. Jam enim non opus erit diviſores tentare niſi qui ſunt inter hos limites.

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## De limitibus æquationum.

190. I. *Radicum affirmativarum limites invenire.*

Minuantur radices æquationis datæ terminis progreſſionis 1, 2, 3, &c. ſucceſſive, donec omnes novæ æquationis termini evadant affirmativi, erunt omnes radices hujus negativæ, adeoque & radices datæ æquationis ſic diminutæ; quarum idcirco decrementum in hoc caſu omnem radicem affirmativam tranſcendet, § 158, ſive erit limes.

Ut ſi proponatur æquatio $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120 = 0$, evadent coefficientes æquationes novæ, per *Reg.* § 157.

| Si $x$ ſit | 1 | 2 | 3 &c. |
|---|---|---|---|
| $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x - 120$ | $-38$ | $+46$ | |
| $5x^4 - 8x^3 - 30x^2 + 60x + 63$ - | $+90$ | $+79$ | |
| $10x^3 - 12x^2 - 30x + 30$ - - - | $-2$ | $+2$ | |
| $10x^2 - 8x - 10$ - - - - - | $-8$ | $+14$ | |
| $5x - 2$ - - - - - - - | $+3$ | $+8$ | |
| 1 - - - - - - - - - - | $+1$ | $+1$ | |

Quoniam igitur ſubſtituendo 1 pro $x$ coefficientes aliquæ prodeunt negativæ, nimirum ultima $= -38$, quarta $= -2$, tertia $= -8$, limes erit major quam 1; at ſubſtituendo 2 pro $x$, prodeunt omnes affirmativæ, unde erit 2 major quam radicum affirmativarum maxima. Et hic ob-

ſervandum eſt, quod ſi coefficiens aliqua evaſerit negativa reliquarum valores invenire non neceſſe erit. Sic in hoc exemplo, ſi ſcribatur 1 pro $x$, prodit tertia coefficiens $10x^2 - 8x - 10 = -8$ negativa, adeoque operationem ulterius proſequi non neceſſe erit. Sæpe quidem ſufficiet ſi ſeligamus coefficientem aliquam ubi termini negativi maxime prævalere videntur, reliquis omiſſis.

191. Atque ideo, ſi termini duo pluresve primi æquationis propoſitæ ſint affirmativi, non neceſſe erit omnes coefficientes novæ æquationis eruere; ſiſti poteſt operatio, ubi ad coefficientem aliquam pervenitur cujus termini omnes ſunt affirmativi præter ultimum. Sic habitâ æquatione $x^7 + 3x^6 + 5x^5 - 7x^4 + 9x^3 - 11x^2 + 13x - 15 = 0$, coefficientes novæ æquationis evadent

$$x^7 + 3x^6 + 5x^5 - 7x^4 + 9x^3 - 11x^2 + 13x - 15$$
$$7x^6 + 18x^5 + 25x^4 - 28x^3 + 27x^2 - 22x + 13$$
$$21x^5 + 45x^4 + 50x^3 - 42x^2 + 27x - 11$$
$$35x^4 + 60x^3 + 50x^2 - 28x + 9$$
$$35x^3 + 45x^2 + 25x - 7$$

&c. &c.

Inventis autem his quinque coefficientibus ultimis æquationis novæ, operationem ultra continuare non opus, cum coefficientis proxime inſequentis $21x^2 + 18x + 5$, necnon reliquarum deinceps, termini omnes ſunt affirmativi.

192. II. *Radicum negativarum limites invenire.*

Augeantur radices datæ æquationis, &c, donec termini alterni tantum evadant negativi, erunt omnes radices datæ

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æquationis ſic auctæ, affirmativæ; quarum idcirco augmentum, &c. erit radicum negativarum limes.

*Vel quod perinde eſt*, mutentur ſigna terminorum alternorum, & inveniatur per *præced.* limes radicum affirmativarum æquationis reſultantis, hic enim mutato ſigno limes erit radicum negativarum datæ æquationis.

Sic mutatis ſignis terminorum alternorum, coefficientes æquationis propoſitæ evadent

$$x^5 + 2x^4 - 10x^3 - 30x^2 + 63x + 120$$
$$5x^4 + 8x^3 - 30x^2 - 60x + 63$$
$$10x^3 + 12x^2 - 30x - 30$$
$$10x^2 + 8x - 10$$
&c.

Ex his ſeligo coefficientem aliquam ubi termini negativi maxime prævalere videntur, puta $5x^4 + 8x^3 - 30x^2 - 60x + 63$, & hic ſubſtituendo pro $x$ numeros 1, & 2, prodeunt numeri negativi — 14 & — 33 reſpective. Unde limes erit major quam — 2. Subſtituendo autem numerum 3 prodit numerus affirmativus 234. Et ſimiliter in cæteris coefficientibus ſubſtituendo numerum 3 pro $x$ prodit ſemper numerus affirmativus. Id quod ex inſpectione ſola colligere licet. Quare numerus — 3 tranſcendit omnes radices negativas. Atque ita habentur limites 2 & — 3 inter quos radices omnes æquationis propoſitæ conſiſtunt. Jam enim non opus erit diviſores termini ultimi 120 tentare, niſi qui ſunt inter hos limites, nimirum diviſores 1, — 1, & — 2. Sed horum nullus radix eſt, unde certum eſt æquationem non habere radicem niſi quæ ſit ſurda.

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193. Hæc limitum inventio uſui erit in reductione æquationum quarum radices inter anguſtos limites concludantur, ſeu ſint parum inæquales inter ſe. Quod ſi radicum una ad maximum termini ultimi diviſorem accedat, altera ad minimum, factores complures ultimi termini inter ipſos limites conſiſtere poſſunt; quo in caſu, parum ſubſidii ad laborem operationis minuendum conferet hæc methodus.

Aliam methodum ſeligendi diviſores ultimi termini ad æquationem dividendam idoneos, *Newtonianâ* in praxi ſæpe commodiorem adjungere libet. Hujus inventorem fuiſſe *Waeſſenaer* refert *Schooten*, *Com. in Geom. Cartes*, *lib.* 3. *p.* 307. *Edit.* 1659.

194. *Minuantur radices æquationis propoſitæ numero quodam dato*, *puta unitate* (unitatem, vel denarium ad id aſſumere commodiſſimum erit) *& radices æquationis novæ unitate minores evadent radicibus datæ. Dein augeantur unitate diviſores ultimi termini novæ æquationis, & ſi horum aliqui ſic aucti inter diviſores termini ultimi datæ æquationis reperiantur, talium præ cæteris ratio habenda eſt. Et rurſus, augeantur unitate radices, &c. Dein minuantur unitate diviſores, &c. Secunda hæc operatio plures adhuc diviſores ſuperfluos reſecabit; & aliquando omnes tollet; ubi nimirum radices nullæ ſint rationales.*

Sic in æquatione $x^3 - x^2 - 30x + 72 = 0$, ſubſtituendo $+1$ pro $x$, fiet ultimus terminus novæ æquationis $+42$, ejuſque diviſores nempe 1, 2, 3, 6, 7, 14, 21, 42, $-1$, $-2$, &c. unitate aucti evadent 2, 3, 4, 7, 8, 15, 22, 43, 0, $-1$, $-2$, $-5$, $-6$, &c. Quibus collatis cum diviſoribus ultimi termini æquationis datæ 72, nempe

1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72, — 1, — 2, &c. utrinque deprehendentur 2, 3, 4, 8, — 1, — 2, — 6. Et rurſus, ſubſtituendo — 1 pro $x$, fiet ultimus terminus novæ æquationis + 100, ejuſque diviſores nempe, 1, 2, 4, 5, 10, 20, 25, 50, 100, — 1, — 2, — 4, &c. unitate diminuti evadent, 1, 3′, 4′, 9, &c. — 2′, — 3, — 5, — 6′, &c. quorum + 3, + 4, — 2, — 6, cum radicibus per præcedentem operationem reſervatis conſentiunt: ergo ultimi termini diviſores ſimplices tentandi, ſunt + 3, + 4, — 2 & — 6, vel æquationis diviſores compoſiti $x - 3$, $x - 4$, &c. unde prodeunt radices + 3, + 4, & — 6.

Sic quoque in æquatione præcedente $x^5 - 2x^4$ &c. — 120 = 0 ſubſtituendo + 1 pro $x$, fit ultimus terminus novæ æquationis — 38, ejuſque diviſores, 1, 2, 19, 38, — 1, — 2, &c. unitate aucti evadent 2, 3, 20, 39, — 1, — 18, — 37, quorum 2, 3, 20, — 1 tantum inter diviſores ultimi termini 120 reperiuntur. Et rurſus, ſubſtituendo — 1 pro $x$, prodit, &c. — 146, ejuſque diviſores 1, 2, 73, 146, — 1, — 2, &c. unitate diminuti evadunt 1, 72, 145, — 2, — 3, &c. omnes nimirum a diviſoribus per præcedentem operationem reſervatis diverſi: unde colligitur æquationem nullam ſortiri radicem niſi quæ ſit ſurda.

Diviſores ultimi termini datæ æquationis per hanc regulam *rejectos*, inter radices datæ æquationis cenſeri non poſſe ex eo conſtat, quod ſi fingantur radices eſſe datæ æquationis, unitate auctæ vel diminutæ deberent eſſe ex

diviſoribus ultimi termini novæ æquationis, contra hypotheſin. Ex diviſoribus quoque *reſervatis*, raro omnes munere radicum fungentur, verùm diviſores hujuſmodi inutiles, ex operatione iteratâ minuendi radices, &c. deprehendi poſſint.

### De inventione divisorum compositorum æquationis.

195. Ex hâc *Waeſſanaeri* methodo ſeligendi diviſores ultimi termini æquationis datæ quorum præ cæteris ratio habenda eſt, deduxiſſe videtur *Newtonus* Regulam præclaram pro *inventione diviſorum compoſitorum unius dimenſionis*. Quam ſic inveſtigare licet.

Finge $ax + b$ diviſorem eſſe æquationis $px^n + qx^{n-1} + rx^{n-2}$ &c. $+ u = 0$, ubi $a$, $b$, $p$, $q$, &c. deſignant numeros integros. Neceſſe eſt diviſoris partem priorem $ax$, atque idcirco coefficientem ejus $a$, eſſe diviſorem numeralem altiſſimi termini $px^n$; itemque partem poſteriorem $b$ eſſe diviſorem ultimi termini $u$: nam ſi non, $ax + b$ totius æquationis non foret diviſor. Subſtitue jam pro $x$ ſigillatim tres vel plures terminos progreſſionis arithmeticæ decreſcentis $3, 2, 1, 0, -1$, &c. et erunt valores reſultantes



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| $x$ | $px^n + qx^{n-1} + rx^{n-2}$ &c. $+ u$ | $ax + b$ |
|---|---|---|
| 2 - | $2^n p + 2^{n-1} q + 2^{n-2} r$ &c. $+ u$ | $2a + b$ |
| 1 - | $p \quad + q \quad + r$ &c. $+ u$ | $a + b$ |
| 0 - | $* \quad * \quad * \quad + u$ | $* + b$ |
| —1 - | $\mp p \quad \pm q \quad \mp r$ &c. $+ u$ | $-a + b$ |
| &c. | &c. | &c. |

ubi obſervandum eſt, quod diviſoris valores reſultantes $2a + b$, $a + b$, $+ b$, &c. conſtituent alteram progreſſionem arithmeticam communi differentia $a$ decreſcentem, cujus terminus quiſque erit diviſor valoris correſpondentis ex ſubſtitutione ejuſdem numeri pro $x$ in æquatione datâ reſultantis; adeoque terminus $b$ e regione zero, diviſor erit valoris correſpondentis $u$, affirmativè ponendus ſi deſcendat ſeries diviſorum, negativè ſi aſcendat, utpote in progreſſione decreſcente conſtitutus. Hinc elicitur regula inſequens.

196. *In æquatione propoſitâ pro* x *ſubſtitue ſigillatim tres vel plures terminos progreſſionis arithmeticæ,* 3, 2, 1, 0, — 1, *&c. ac valores totidem reſultantes una cum omnibus ſuis diviſoribus tam affirmativis quam negativis, ſtatue e regione correſpondentium terminorum progreſſionis. Dein e regione etiam ſtatue progreſſiones arithmeticas decreſcentes quæ per omnium numerorum diviſores percurrunt, & quarum termini differunt vel unitate, vel numero aliquo qui dividit altiſſimum terminum propoſitæ æquationis. Si qua occurrit ejuſmodi progreſſio, iſte terminus ejus qui ſtat e regione zero in progreſſione prima, cum ſigno ſuo annexus quantitati incognitæ* x *in communem differentiam terminorum ductæ, componet quantitatem per quam diviſio tentanda eſt.*

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Ut ſi æquatio ſit $x^3 - x^2 - 10x + 6 = 0$, pro $x$ ſubſtituendo ſigillatim terminos progreſſionis 1, 0, − 1, orientur numeri − 4, + 6, + 14, quos cum omnibus eorum diviſoribus colloca è regione terminorum progreſſionis, hoc modo.

$$x = \left\{\begin{array}{r} +1 \\ 0 \\ -1 \end{array}\right\} x^3 - x^2 - 10x + 6 = \left\{\begin{array}{r} -4 \\ +6 \\ +14 \end{array}\right\} \begin{array}{l} \text{Diviſores} \\ 1.2.4. \\ 1.2.3.6 \\ 1.2.7.14 \end{array} \left\{\begin{array}{r} \text{Progr. ar.} \\ +4 \\ +3 \\ +2 \end{array}\right.$$

Dein quoniam altiſſimus terminus $x^3$ per nullum numerorum præter unitatem diviſibilis eſt, quære in diviſoribus progreſſionem decreſcentem cujus termini differunt unitate, & hujuſmodi progreſſionis unice reſultantis 4, 3, 2, ſelige terminum + 3, qui ſtat e regione zero in progreſſione prima, & diviſionem tenta per $x + 3$, unde prodibit $x^2 - 4x + 2 = 0$ cujus radices ſunt $2 \pm \sqrt{2}$. Adeoque tres radices ſunt $-3$, $2 + \sqrt{2}$, $2 - \sqrt{2}$.

Et rurſus ſi æquatio ſit $25x^4 + 110x^3 - 161x^2 - 190x + 144 = 0$, § 94, erit

$$x = \left.\begin{array}{r} +1 \\ 0 \\ -1 \\ \&c. \end{array}\right\} 25x^4 + 110x^3\ \&c. \left\{\begin{array}{r} -272 \\ +144 \\ +288 \\ \&c. \end{array}\right\} \begin{array}{l} 1, 2, 4, 8, 16, \&c. \\ 1, 2, 3, 4, 6, \&c. \\ 1, 2, 3, 4, 6, \&c. \\ \&c. \end{array} \left.\right\} \begin{array}{l} +4 +8\&c. -1\&c. \\ +3 +3\&c. -2\&c. \\ +2 -2\&c. -3\&c. \\ \&c. \end{array}$$

Ex his progreſſionibus tres ſeliguntur diviſores tentandi, nempe $x + 3$, $5x + 3$, & $x - 2$, quorum ultimus ſolus ſuccedit, et quotientem dat $25x^3 + 160x^2 + 159x - 72 = 0$.

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Atque ita ſi æquatio ſit $24a^5 - 50a^4 + 49a^3 - 140a^2 + 64a + 30 = 0$, operatio erit ut ſequitur.

| $a =$ | | $24a^5 - 50a^4$ &c. $=$ | | | |
|---|---|---|---|---|---|
| | $+2$ | | 42 | 1, 2, 3, 6, 7, &c. | $+3+3+7$ |
| | $+1$ | | 23 | 1, 23, | $+1-1+1$ |
| | $0$ | | 30 | 1, 2, 3, 5, 6, &c. | $-1-5-5$ |
| | $-1$ | | 297 | 1, 3, 9, 11, 27, &c. | $-3-9-11$ |

Tres occurrunt hic progreſſiones quæ tres dant diviſores tentandos $2a - 1$, $4a - 5$, & $6a - 5$, quorum ultimus ſolus ſuccedit, prodeunte $4a^4 - 5a^3 + 4a^2 - 20a - 6 = 0$.

Atque hic obſervandum, quod progreſſionem 2, 1, 0, $-1$, &c. continuando, facilius detegentur diviſores inutiles. Sic in exemplo noviſſimo, ſi quintus accedat terminus progreſſionis ſubſtituendo $-2$ pro $a$, prodit $-2618$, cujus ex diviſoribus 1, 2, 7, 11, 17, &c. $-17$ ultimam progreſſionem 7, 1, $-5$, $-11$, continuabit, cæteris interruptis.

198. Si nullus occurrit hac methodo diviſor, vel nullus qui dividit propoſitam æquationem, concludendum erit æquationem illam non admittere diviſorem *unius* dimenſionis. Poteſt tamen fortaſſe, ſi plurium ſit quam trium dimenſionum, diviſorem admittere *duarum*. Et ſi ita, diviſor ille eruetur ope methodi inſequentis, quam ſic inveſtigare licet.

Finge $ax^2 + bx + c$ diviſorem eſſe æquationis $px^n + qx^{n-1}$ &c. $+ u = 0$. Neceſſe eſt ut $a$ ſit diviſor numeralis termini altiſſimi $px^n$, aliter enim totam æquatio-

nem non divideret. Subſtitue jam ſigillatim pro $x$ quatuor vel plures terminos progreſſionis arithmeticæ 3, 2, 1, 0, — 1, — 2, et erunt valores reſultantes

| $x$. | $px^n + qx^{n-1} + rx^{n-2} + \&c. + u$ | $ax^2 + bx + c$ |
|---|---|---|
| 3 | $3^n p + 3^{n-1} q + 3^{n-2} r + \&c. + u$ | $9a + 3b + c$ |
| 2 | $2^n p + 2^{n-1} q + 2^{n-2} r + \&c. + u$ | $4a + 2b + c$ |
| 1 | $p + q + r + \&c. + u$ | $a + b + c$ |
| 0 | $*\ \ *\ \ *\ \ \&c. + u$ | $*\ \ * + c$ |
| —1 | $\mp p \pm q \mp r \pm \&c. + u$ | $a - b + c$ |
| —2 | $\mp 2^n p \pm 2^{n-1} q \mp 2^{n-2} r \pm \&c. + u$ | $4a - 2b + c$ |

Ubi obſervandum eſt, quod ſi de valoribus ſucceſſivis diviſoris generalis ſubducantur primi horum termini $9a$, $4a$, $a$, &c. reſpective, reſidua $3b + c$, $2b + c$, $b + c$, &c. conſtituent alteram progreſſionem arithmeticam cujus differentia communis eſt $b$, & cujus terminus $c$ e regione zerò in progreſſione prima erit ex diviſoribus termini correſpondentis $u$. Unde elicitur regula inſequens.

198. *In æquatione propoſita ſubſtitue pro* x, *quatuor vel plures terminos progreſſionis hujus* 3, 2, 1, 0, — 1, — 2, *& de diviſoribus omnibus numerorum reſultantium tam affirmativis quam negativis, ſubduc quadrata correſpondentium terminorum progreſſionis illius, in diviſorem aliquem numeralem termini altiſſimi æquationis ducta; & reſidua propriis cum ſignis e regione progreſſionis colloca. Dein progreſſiones omnes collaterales tam creſcentes quam decreſcentes nota quæ per iſta reſidua percurrunt. Et*

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*erit diviſor ille duarum dimenſionum tentandus, cujus primi termini coefficiens eſt prædictus diviſor numeralis termini altiſſimi; ſecundi coefficiens, differentia communis progreſſionis illius collateralis cum ſigno affirmativo ſi decreſcat progreſſio, negativo ſi creſcat; & e tertius terminus, iſte progreſſionis collateralis terminus qui ſtat regione zero in progreſſione primâ.*

*Invenietur autem differentia communis, ſubducendo terminum iſtiuſmodi progreſſionis e regione zero in progreſſione prima, de termino ejuſdem progreſſionis proxime ſuperiore, eritque differentia affirmativa aut negativa prout progreſſio decreſcat vel creſcat.*

Ut ſi æquatio propoſita ſit $x^4 - x^3 - 5x^2 + 12x - 6 = 0$, ſubſtituendo pro $x$, ſucceſſive 3, 2, 1, 0, — 1, — 2, numeros prodeuntes 39, 6, 1, — 6, — 21, — 26, una cum eorum diviſoribus tam affirmativis quam negativis ſic diſpone

| | | | | | |
|---|---|---|---|---|---|
| 3 | 39 | 1,3,13,39,—1,—3,&c. | —8,—6,4,30,—10,—12,&c. | +4 | —6 |
| 2 | 6 | 1,2, 3, 6,—1,—2,&c. | —5,—2,—1,2,—5,— 6,&c. | +2 | —3 |
| 1 | 1 | 1, —1, | 0,—2, | +0 | +0 |
| 0 | —6 | 1,2, 3, 6,—1,—2,&c. | 1, 2, 3, 6,—1,—3,&c. | —2 | +3 |
| —1 | —21 | 1,3, 7,21,—1,—3,&c. | 0, 2, 6,20,—2,—4,&c. | —4 | +6 |
| —2 | —26 | 1,2,13,26,—1,—2,&c. | —3,—2, 9,22,—5,—6,&c. | —6 | +9 |

Dein de his diviſoribus ſigillatim ſubduc quadrata terminorum progreſſionis primæ ducta in diviſorem numeralem termini $x^4$, qui unitas eſt, *viz.* terminos 9, 4, 1, 0, 1, 4, &c. prodibunt termini — 8, — 6, &c. — 3, — 2, &c. e latere pariter diſpoſiti; qui duas præbent progreſſiones arithmeticas + 4 + 2, + 0, &c. et — 6, — 3, + 0, &c.

collaterales. Unde prodeunt diviſores tentandi $x^2 + 2x - 2$ & $x^2 - 3x + 3$, per quorum utrumque res ſuccedit. At factoris $x^2 + 2x - 2 = 0$ radices ſunt $-1 \pm \sqrt{-3}$, irrationales, et factoris $x^2 - 3x + 3 = 0$ radices $\frac{+3 \pm \sqrt{-3}}{2}$ ſunt impoſſibiles. Hæc regula igitur aliquando utilis erit in radicibus æquationum ſurdis & impoſſibilibus eruendis.

199. Si nullus inveniri poteſt hoc pacto diviſor qui ſuccedit, concludendum eſt æquationem propoſitam non admittere diviſorem duarum dimenſionum. Hinc quoque concludere licet, æquationem non admittere diviſorem trium dimenſionum, ſi modo quintum non tranſcendat gradum. Sin æquatio ad ſextum gradum aut ultra aſſurgat, poteſt forſan nullum admittere diviſorem niſi qui ſit trium pluriumve dimenſionum. Poſſet quidem eadem methodus extendi ad inventionem hujuſmodi diviſorum quærendo in prædictis reſiduis, progreſſiones non arithmeticas, ſed alias quaſdam quarum terminorum differentiæ primæ, ſecundæ, tertiæ, &c. ſunt in arithmetica progreſſione: at in his tyro non eſt detinendus. Siquis tamen hæc deſiderat, adeat *M^c Laurin*, *Algeb. Part.* II. § 70.

## De inventione diviſorum æquationum literalium quarum termini ſunt homogenei.

200. Proponatur æquatio $x^n - ax^{n-1} - 5a^2x^{n-2} + 12a^3x^{n-3}$ &c. $- 6a^n = 0$. Et ſi dividantur radices

hujus per $a$, ſubſtituendo $ay$ pro $x$ emerget $y^n - y^{n-1} - 5y^{n-2} + 12y^{n-3}$ &c. $- 6 = 0$. Deinde per regulas præcedentes quære hujus diviſorem, & reſtituendo in diviſore prodeunte $\frac{x}{a}$ pro $y$, & tollendo fractiones, habebitur datæ æquationis diviſor. Hinc elicitur regula inſequens.

201. *Ubi in æquatione propoſita duæ ſunt literæ, & omnes ejus termini ad dimenſiones æque altas aſcendunt; pro una iſtarum literarum pone unitatem, dein per regulas præcedentes quære diviſorem, ac diviſoris hujus comple deficientes dimenſiones reſtituendo literam illam pro unitate.*

Ut ſi æquatio ſit $6x^4 - ax^3 - 21a^2x^2 + 3a^3x + 20a^4 = 0$ ponendo 1 pro $a$, evadit $6x^4 - x^3 - 21x^2 + 3x + 20 = 0$, cujus eruetur diviſor $3x + 4$, & completa deficiente dimenſione poſterioris termini per dimenſionem $a$, ſit $3x + 4a$ diviſor quæſitus. Ita ſi æquatio ſit $x^4 - bx^3 - 5b^2x^2 + 12b^3x - 6b^4 = 0$ poſito 1 pro $b$, & æquationis reſultantis $x^4 - x^3 - 5x^2 + 12x - 6 = 0$ invento diviſore $x^2 + 2x - 2$, completis ejus deficientibus dimenſionibus per dimenſiones $b$, habebitur diviſor quæſitus $x^2 + 2bx - 2b^2$.

Poſſet eadem methodus extendi ad inventionem diviſorum ubi in æquatione propoſitâ tres vel plures ſunt literæ, idque ſive termini ſint æquialti ſive non; cæterum hiſce non eſt immorandum. Conſulat lector horum ſtudioſus *Clairaut*, *Algeb.* 4ta edit. 1768, p. 184, &c.

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## De resolutione æquationum duas pluresve radices inter se æquales habentium.

202. Ubi ex § 183, vel aliunde compertum eſt æquationem binas pluresve radices æquales ſortiri, conferendo hanc cum æquatione dimenſionis inferioris inde eliciendâ, per § 185, facile inveniri poteſt valor unius radicum æqualium.

203. Ut ſi proponatur æquatio cubica $x^3 - px^2 + qx - r = 0$, duas habens radices inter ſe æquales, inde elicietur per § 185, æquatio quadratica $3x^2 - 2px + q = 0$, radicem habens unam radicum æqualium. Collatis autem hiſce æquationibus inter ſe, per Reg. § 70, reſultabit quadratica $px^2 - 2qx + 3r = 0$, et quadraticis iterum collatis, reſultabit ſimplex æq. $(2p^2 - 6q) x - pq + 9r = 0$, unde prodit $x = \frac{pq - 9r}{2p^2 - 6q}$.

Hunc valorem $x$ unam radicum æqualium exponere, per compoſitionem æquationum conſtabit. Sint enim radices $a, a, b$, erit $p = 2a + b$, $q = a^2 + 2ab$, & $r = a^2b$. Adeoque his ſubſtitutis, prodit $x = \frac{pq - 9r}{2p^2 - 6q} = \frac{2a^3 - 4a^2b + 2ab^2}{2a^2 - 4ab + 2b^2} = a$ *. Sic habitâ æquatione $x^3 + 5x^2 - 32x + 36 = 0$,

* Vide *Diſſert.* D. *Hellins*, *Phil. Tranſ.* an. 1782, p. 417.

fiat $p = -5$, $q = -32$, $r = -36$, erit $x = \frac{160 + 324}{50 + 192}$ $= \frac{284}{242} = 2$. Adeoque dividendo æquationem datam per $(x - 2)^2$ eruetur quotiens $x + 9 = 0$. Ergo tertia radix erit $-9$. Habitâ quoque æquatione $x^3 - \frac{3}{4}x - \frac{1}{4} = 0$, erit $x = \frac{-9r}{-6q} = \frac{-\frac{9}{4}}{\frac{18}{4}} = -\frac{1}{2}$. Adeoque tertia radix $+1$.

204. Si proponatur æquatio biquadratica $x^4 - px^3 + qx^2 - rx + s = 0$, duas habens radices reales, harum una erit radix æquationis cubicæ $4x^3 - 3px^2 + 2qx - r = 0$. Et collatis his æquationibus, emerget cubica $px^3 - 2qx^2 + 3rx - 4s = 0$. Et collatis cubicis, emerget quadratica $(3p^2 - 8q)\,x^2 + (12r - 2pq)\,x + pr - 16s = 0$, ſeu abbreviandi gratiâ, $Ax^2 + Bx + C = 0$. Et collatis iterum cubicis, per Reg. § 73, emerget altera quadratica $(pr - 16s)\,x^2 + (12ps - 2qr)\,x + 3r^2 - 8qs = 0$, ſeu $Cx^2 + Dx + E = 0$. Dein collatis quadraticis, emerget $(AD - BC)\,x + AE - C^2 = 0$, adeoque $x = \frac{C^2 - AE}{AD - BC}$. Reſtitutis autem valoribus A, B, C, &c. prodit

$$x = \frac{6p^2qs - 2p^2r^2 - 8prs + 6qr^2 - 16q^2s + 64s^2}{9p^3s - p^2qr - 3pr^2 - 32pqs + 4q^2r + 48rs}.$$

Deleto autem termino ſecundo æquationis, erit $x = \frac{6qr^2 - 16q^2s + 64s^2}{4q^2r + 48rs}$ $= \frac{(3r^2 - 8qs)\,q + 32s^2}{(q^2 + 12s)\,2r}$.

Sic habita æquatione $x^4 - 11x^2 + 18x - 8 = 0$, ſtatuendo $q = -11$, $r = -18$, $s = -8$, prodit $x = \frac{-(972 - 704)\,11 + 2048}{-(121 - 96)\,36} = \frac{-900}{-900} = +1$. Et dividendo æquationem per $(x-1)^2$ prodit quotiens $x^2 + 2x - 8 = 0$, cujus reductio dabit reliquas radices $+2$ & $-4$.

Si tres biquadraticæ radices ſint inter ſe æquales, harum duæ erunt radices æquationis cubicæ $4x^3 - 3px^2 + 2qx - r = 0$, et una, quadraticæ $6x^2 - 3px + q = 0$. Et his collatis, eruetur $x = \frac{pq - 6r}{3p^2 - 8q}$. Ut ſi æquatio ſit $x^4 - x^3 - 3x^2 + 5x - 2 = 0$, erit $x = \frac{-3 + 30}{3 + 24} = +1$, adeoque dividendo per $(x-1)^3$ eruetur quotiens $x + 2 = 0$, adeoque quarta radix erit $-2$.

Iiſdem veſtigiis inſiſtendo valores radicum æqualium, & inde reliquarum, in æquationibus dimenſionum ſuperiorum eruere licet.

205. Hactenus reductionem æquationum tradidimus quæ rationales diviſores admittant. Deficientibus autem hujuſmodi diviſoribus, aliquando reduci poteſt æquatio quatuor, ſex aut plurium dimenſionum, per ſurdum aliquem diviſorem. Methodum quidem tales diviſores inveniendi tradit *Newtonus*, quærendo nimirum quantitatem aliquam complexam quæ quadratum utriuſque partis æquationis propoſitæ compleat, quâ additâ, per extractionem

ſurdæ radicis quadraticæ utrobique res ſuccedit. Has autem reductiones prætereo, quarum utpote raro poſſibilium, perexiguus ſit uſus, ipſo fatente *Newtono*, " qui rei poſſi-" bilitatem potius quam praxim commodiſſimam voluerit " exponere." De his conſulendus eſt *M<sup>c</sup>Laurin*, &c. et præcipue *Simpſon*, *Mathematical Tracts*.

Hanc analyſeos partem haud minore ſtudio excoluiſſe videtur *Leibnitz*, *Newtoni* æmulus, licet meditationes ejus in lucem non prodierunt. In epiſtolis ſuis ad *Oldenburg*, an. 1676, ita ſcribit.

" De æquationum radicibus ſurdis generalibus inveni-" endis, ſive quod idem eſt, tollendis æquationum poteſ-" tatibus intermediis, multa & ego meditatus ſum; & " jam vere anni ſuperioris ſpecimina *Hugenio* communica-" veram regularum *Cardanicis* ſimilium. Seriem enim ha-" bebam ejuſmodi regularum in infinitum euntem, in " quibus & *Cardanica* continebatur. Sed ultra gradum " cubicum non erant generales. Perſpexi tamen inde me-" thodum veram progrediendi longius.——Sed ſiquis la-" borem non ſubterfugeret, eum docere poſſum methodum " analyticam generalem infallibilem, per quam omnium " æquationum radices generales invenire liceret." Vide *Commerc. Epiſtol.* N° 53.

Merito tamen dubitare licet, annon promiſſa tam ingentia celeberrimi ſcriptoris fiduciam fefelliſſent, ſimodo calculis ſuis ultimum manum imponere voluerit. Multa enim ineunte calculo facilia fore fidimus, quæ temere aſſumpta & prorſus impoſſibilia arguet matura inveſtigatio.

Pergamus jam æquationum cubicarum & biquadraticarum reſolutiones generaliores docere.

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## De radicibus æquationum cubicarum.

206. Quo melius inſtituatur diſquiſitio de radicum cubicarum natura, commodum erit æquationem quamcunque cubicam, delendo terminum ſecundum, ad formam ſimpliciorem reducere $x^3 * . qx = . r$; ubi variantibus ſignis $q$ & $r$, quatuor diſtinguuntur caſus, quorum tres priores tantùm (ut in quadraticis) pro legitimis habuerunt analyſtæ vetuſtiores.

$$\text{I.} \quad x^3 + qx = + r$$
$$\text{II.} \quad x^3 - qx = + r$$
$$\text{III.} \quad x^3 - qx = - r$$
$$\text{IV.} \quad x^3 + qx = - r.$$

207. Deſignante $p$, primam radicem quoquo modo inveniendam, quæ ſemper realis erit, dividatur æquatio $x^3 + qx - r = 0$, per factorem ſimplicem $x - p = 0$, et (evaneſcente reſiduo $p^3 + pq - r$) emergit quotiens $x^2 + px + p^2 + q = 0$. Et extracta hujus radice quadratica, prodit $x = -\frac{1}{2}p \pm \sqrt{-\frac{3}{4}p^2 - q}$. Æquationis itaque $x^3 + qx = r$, tres radices ſunt $p$, $-\frac{1}{2}p + \sqrt{-\frac{3}{4}p^2 - q}$, vel $-\frac{1}{2}p - \sqrt{-\frac{3}{4}p^2 - q}$. *Et ſic univerſaliter: æquationis* $x^3 . qx = r$, *ſi una radix ſit* $p$, *erunt reliquæ* $\frac{1}{2}p \pm \sqrt{-\frac{3}{4}p^2 . q}$, *ubi* $\frac{1}{2}p$ & $q$ *cum ſignis mutatis, ſed* $\frac{3}{4}p^2$ *ſemper negativè ponendum.* Sunt igitur duæ

poſteriores radices ſemper impoſſibiles ubi $q$ in data æquatione affirmativum eſt, adeoque $\sqrt{-\frac{3}{4}p^2 - q}$ quantitas impoſſibilis; aut ubi $q$ negativum eſt & ſimul minus quam $\frac{3}{4}p^2$, adeoque $\sqrt{-\frac{3}{4}p^2 + q}$ impoſſibilis: at ubi $q$ negativum eſt ac ſimul æquale aut majus quam $\frac{3}{4}p^2$, ſunt ſimul reales, aut evaneſcente, aut poſſibili jam factâ $\sqrt{-\frac{3}{4}p^2 + q}$.

Ibi itaque verſabitur limes radicum realium & impoſſibilium ubi $q$ in data æquatione negativum eſt & æquari primùm incipit $\frac{3}{4}p^2$. In ipſo autem limite, prodit $\overline{\frac{1}{3}q}|^3 = \overline{\frac{1}{4}p^2}|^3 = \frac{1}{64}p^6$. Itemque ſubſtituendo $\frac{3}{4}p^2$ pro $q$ in æquationem $p^3 - pq = \cdot\, r$, prodit $\frac{1}{4}p^3 = \cdot\, r$, adeoque $\overline{\frac{1}{2}r}|^2 = \overline{\frac{1}{8}p^3}|^2 = \frac{1}{64}p^6$. Ergo, $\overline{\frac{1}{3}q}|^3 = (\frac{1}{64}p^6 =)\ \overline{\frac{1}{2}r}|^2$.

Ubi vero $q$ in data æquatione negativum eſt ac ſimul minus quam $\frac{3}{4}p^2$, prodit $\overline{\frac{1}{3}q}|^3$ minus quam $\overline{\frac{1}{4}p^2}|^3$ ſeu quam $\frac{1}{64}p^6$. Sed in hoc caſu, $r\ (= p^3 - pq)$ fit majus quam $\frac{1}{4}p^3$, adeoque $\overline{\frac{1}{2}r}|^2$ majus quam $\overline{\frac{1}{8}p^3}|^2$ ſeu quam $\frac{1}{64}p^6$. Ergo, *a fortiori*, prodit $\overline{\frac{1}{3}q}|^3$ minus quam $\overline{\frac{1}{2}r}|^2$. Contra ſi $q$ ſit majus, &c.

268. Hinc partim ex ſigno quo afficitur $q$ in data æquatione, partim ex magnitudine ejus reſpectu $r$, innoteſcit geneſis radicum realium & impoſſibilium in æquatione $x^3 \cdot qx = \cdot\, r$. Nam ubi $q$ affirmativum eſt, qualiſcunque

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ſit ejus magnitudo, ſemper duæ ex radicibus æquationis ſunt impoſſibiles; idem quoque fiet, ubi $q$ negativum eſt & $\frac{q^3}{27}$ ſimul minus quam $\frac{1}{4}r^2$. At ubi $q$ negativum eſt & $\frac{q^3}{27}$ ſimul æquale aut majus quam $\frac{1}{4}r^2$, ſunt omnes radices reales. Sicut aliunde collegimus, § 179, 183.

Hanc methodum deducendi affectiones radicum æquationum cubicarum termino ſecundo carentium immediate ex ipſa æquationum naturâ, ſuggeſſit diſſertatio viri in hiſce verſatiſſimi, *Hutton*, *on Cubic Æquations*, & *infinite Series* *.

## De resolutione æquationum cubicarum.

209. Proponatur æquatio cubica cujus ſecundus terminus deeſt, $x^3 * + qx = r$. Ad hanc enim formam æquationem omnem cubicam reduci poſſe conſtat ex præcedentibus. Finge $x$ duas in partes dividi, quarum ſit major $y$ & minor $z$. Et ſubſtituendo $y + z$ pro $x$ in hac æquatione, prodit $y^3 + z^3 + 3yz\ (y + z) + q\ (y + z) = r$. Finge iterum $3yz$ æquari $q$ cum ſigno mutato, et terminis $3yz\ (y + z)$ atque $q\ (y + z)$ hoc pacto ſe deſtruentibus, manebit $y^3 + z^3 = r$, ſive, ſubſtituendo $\frac{-q^3}{27y^3}$ pro $z^3$, (ob $3yz$

* *Phil. Tranſ.* an. 1780, p. 387.



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$= - q$, adeoque $z = \frac{-q}{3y}$) erit $y^3 - \frac{q^3}{27y^3} = r$, ſeu $y^6 = ry^3 + \frac{q^3}{27}$. Adeoque per reductionem, $y^3 = \frac{1}{2}r + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ & $y = \sqrt[3]{:\ \frac{1}{2}r + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}$. Itemque $z^3 =$ ($r - y^3 = r - \frac{1}{2}r - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} =$) $\frac{1}{2}r - \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$ & $z = \sqrt[3]{:\ \frac{1}{2}r - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}$ *. *Et ſic univerſaliter: Si ſit* $x^3 \,.\, qx = \,.\, r$, *erit* $x = \sqrt[3]{:\ \frac{1}{2}r + \sqrt{\frac{r^2}{4} \,.\, \frac{q^3}{27}}} + \sqrt[3]{:\ \frac{1}{2}r - \sqrt{\frac{r^2}{4} \,.\, \frac{q^3}{27}}}$. *Ubi* $\frac{1}{2}r$ & $\frac{q^3}{27}$ *iiſdem ſignis ac* r & q *in æquatione priore afficienda ſunt, ſed* $\frac{1}{4}r^2$ *ſemper affirmativè ponendum.*

* Hæc formula appellatur *Cardani regula*, licet inventionem ejus *Scipioni Ferreo* Bononienſi tribuit ipſe *Cardanus*. Diſcipulo ſuo *Antonio M. Florido* formulam tradidit *Ferreus*. At *Florido* cum in certamen veniſſet cum *Nicoluo Tartalea* Brixellenſi, occaſionem dedit ut *Tartalea*, rem diu ſecum meditatus extuderit, cæteroſque formulæ caſus ſimul invenerit: et adverſarii problematis 30 ſpatio duarum horarum (ut ipſe ſcribit) reſolutis, alia viciſſim iſti objecit ex cæteris caſibus, quibus ſolvendis omnino impar erat. Formulam ipſam *Cardano* roganti, multis precibus exoratus impertivit *Tartalea*, ſuppreſsâ tamen demonſtratione, eumque juramento adſtrinxit, ſe hanc neutiquam vulgaturum nec niſi cifris commiſſurum. Verùm *Cardanus* promiſſorum immemor, hanc publici juris fecit in opere quodam *De Arte magna*, an. 1545. Unde ingens controverſia exarſit, reclamante *Tartalea* fidem fregiſſe *Cardanum*, regerente *Cardano*, ſe regulam novis inventis auctam & demonſtratione inſuper munitam, quaſi propriam jure edidiſſe, necnon ſcrupulum injiciente an *Tartalea* inventum ſibi recte vindicaverit. Nec extinctæ ſunt lites niſi cum ipſo *Tartalea*, qui an. 1557 obiit.

Quanti hanc formulam fecit *Cardanus*, ex ipſius verbis conſtabit. " Temporibus noſtris *Scipio Ferreus* Bononienſis capitulum cubi & rerum numero æqualium invenit. Rem ſane pulchram & admirabilem, cum omnem humanam ſubtilitatem, omnis ingenii mortalis claritatem ars hæc longe ſuperet,

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Inventâ primâ radice $x = y + z$, innoteſcent reliquæ. Quoniam enim unitati competit radix cubica triplex, nempe,

" donum profecto cœleſte, experimentum autem virtutis animorum, atque adeo " illuſtre, ut qui hæc attigerit nihil non intelligere poſſe ſe credat." *Cardan. Opera.* Tom IV, p 222. *Cardanum* vero fefellit radices reales, in *Caſu irreducibili*, ſub ſpecie impoſſibili adumbrari. Primus hoc exploratum habuit *Bombellius.*

Hujus regulæ hiſtoriam concludemus cum verſibus Italicis quibus primi caſus ſolutionem expoſuit *Tartalea.*

*Quando che il cubo con le coſe appreſſo*
*S' agguaglia a qualche numero diſcreto,*
*Trovami due altri differenti in eſſo*
*Dapoi terrai queſto per conſueto*
*Ch' il lor producto ſempre ſi eguale*
*Al terzo cubo delle coſe netto ;*
*El reſiduo poi tuo generale*
*Delli lor lati cubi ben ſottratto*
*Verra la tua coſa principale.*

Hoſce verſus ſic iterpretari licet.

Quando *cubus* cum *rebus* adjectis æquatur *numero*, i. e. $x^3 + qx = r$, inveniantur duo alii numeri ($y$ & $z$) quorum differentia ſit $r$, & productum ($yz$) æquale cubo tertiæ partis coefficientis rerum $\left(\frac{q^3}{27}\right)$. Sed per primam conditionem, prodit $r + z = y$ & $y - r = z$, adeoque per ſecundam conditionem, $y^3 - ry = \frac{q^3}{27}$, & per reductionem, $y = \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} + \frac{1}{2}r$. Itemque, $rz + z^2 = \frac{q^3}{27}$, & per reductionem, $z = \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} - \frac{1}{2}r$. Deinde radicibus $y$ & $z$ cu-

1, $\frac{-1+\sqrt{-3}}{2}$, vel $\frac{-1-\sqrt{-3}}{2}$, coefficiente triplici exponendæ ſunt $y$ & $z$, ſcilicet

$$1\,y, \text{ vel } \frac{-1+\sqrt{-3}}{2}y, \text{ vel } \frac{-1-\sqrt{-3}}{2}y,$$

$$\&\ 1\,z, \text{ vel } \frac{-1-\sqrt{-3}}{2}z, \text{ vel } \frac{-1+\sqrt{-3}}{2}z.$$

Ubi $\sqrt{-3}$ contrariis ſignis afficiendum in valoribus correſpondentibus $y$ & $z$, ut productum ex his in unoquoque caſu ſit $yz = \frac{-q}{3}$. Quod hoc pacto ita ſe habet, nam $\frac{-1+\sqrt{-3}}{2}y \times \frac{-1-\sqrt{-3}}{2}z = \frac{1+3}{4}yz = yz.$

Prodeunt igitur tres radices,

1. $y+z$,

2. $\frac{-1+\sqrt{-3}}{2}y + \frac{-1-\sqrt{-3}}{2}z =$

$$= \frac{-y-z}{2} + \frac{y-z}{2}\sqrt{-3} = -\frac{y+z}{2} + \frac{y-z}{2}\sqrt{-3}.$$

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bice extractis, & ſubductâ majore radice cubicâ a minore, prodit *res* ſeu $x = \sqrt[3]{:}\ \sqrt{\frac{r^2}{4}+\frac{q^3}{27}} + \frac{1}{2}r - \sqrt[3]{:}\ \sqrt{\frac{r^2}{4}+\frac{q^3}{27}} - \frac{1}{2}r$, ſeu quod eodem redit, $= \sqrt[3]{:}\ \frac{1}{2}r + \sqrt{\frac{r^2}{4}+\frac{q^3}{27}} + \sqrt[3]{:}\ \frac{1}{2}r - \sqrt{\frac{r^2}{4}+\frac{q^3}{27}}$. Vide *Montucla, Hiſt. de Mathem.* Vol. I, p. 480. Et *Diſſert. De Gua. Mem. de l'Acad.* an. 1741.

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3. $\frac{-1-\sqrt{-3}}{2}y + \frac{-1+\sqrt{-3}}{2}z =$

$$= \frac{-y-z}{2} + \frac{-y+z}{2}\sqrt{-3} = -\frac{y+z}{2} - \frac{y-z}{2}\sqrt{-3}.$$

Ubi obſervandum eſt, quod ſi $q$ ſit negativum & $\frac{q^3}{27}$ ſimul æquale $\frac{1}{4}r^2$, evaneſcente $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}}$ fiet tam $y$ quam $z$ $= \sqrt[3]{:\frac{1}{2}r}$, adeoque prima radix $y + z = 2\sqrt[3]{\frac{1}{2}r}$, reliquæ autem, evaneſcente $\frac{y-z}{2}\sqrt{-3}$, evadent $-\frac{y+z}{2} = -y$ reſpective, adeoque coaleſcent in $-\sqrt[3]{\frac{1}{2}r}$.

210. Et hoc modo erui poſſunt radices omnium cubicarum æquationum, ubi $q$ affirmativum eſt; vel ubi $q$ negativum eſt & $\frac{q^3}{27}$ ſimul minus quam $\frac{1}{4}r^2$, adeoque $\sqrt{\frac{1}{4}r^2 - \frac{q^3}{27}}$ quantitas realis, id eſt, ubi duæ ex radicibus æquationis ſunt impoſſibiles; vel etiam ubi $q$ negativum eſt & $\frac{q^3}{27}$ ſimul æquale $\frac{1}{4}r^2$, id eſt, ubi duæ ex radicibus ſunt equales. At ubi $q$ negativum eſt & $\frac{q^3}{27}$ ſimul majus quam $\frac{1}{4}r^2$, fit $\sqrt{\frac{1}{4}r^2 - \frac{q^3}{27}}$ quantitas impoſſibilis, adeoque tres radices ſpeciem ementiuntur impoſſibilem, ubi revera ſunt omnes reales. Vide § 208.

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211. Hic caſus, qui vulgo appellatur *irreducibilis*, plus quam 200 annos analyſtarum ingenia exercuit. De hoc ita *Schootenus* *, " Ubi $\frac{1}{4}r^2$ ſit minus quam $\frac{-q^3}{27}$ ibi præ-
" dicta regula non habet locum, nec ejuſdem beneficio
" radix ullo modo intelligibili explicari poteſt. Quæ qui-
" dem res olim multæ fuit caliginis, & ut ſcribit *Albertus*
" *Girardus* in libello cui titulus, *Invention nouvelle en l'Al-*
" *gebra*, qui anno 1629 prodiit: *Hoc eſt in quo auctores hac-*
" *tenus fuerunt valde intricati, & ut verum fatear, in re quam*
" *maxime difficili.*" Nec minus in hoc etiamnum hærent Analyſtæ quam in circuli quadraturâ Geometræ †. Latet certe vitium in altera conditionum ad reſolutionem datæ æquationis adſcitarum. Nam licet quantitatem $x$ duas in partes $y$ & $z$ dividere *ſemper* poſſibile ſit, tamen partium rectangulum $yz$ æquari $\frac{1}{3}q$ cum ſigno mutato *aliquando* impoſſibile erit. Ubi quidem $q$ affirmativum eſt, adeoque $yz$ $(= -\frac{1}{3}q)$ negativum, contraria ſortientur ſigna $y$ & $z$, atque idcirco creſcente minore quantitate $z$ a nihilo ad infinitum, ſi ita ſimul creſcat $y$ ut exceſſus ejus ſupra $z$ ſit ſemper æqualis $x$, creſcet quoque rectangulum $-yz$ a nihilo ad infinitum, atque ideo alicubi inter creſcendum fiet æquale $-\frac{1}{3}q$. Ubi vero $q$ negativum eſt, evadet $yz$ $(= -\frac{1}{3}q)$ affirmativum, ſignis igitur ſimilibus afficientur $y$ & $z$, affirmativis utique ſi $x$ affirmativum ſit, negativis ſi ſecus. Cæterum $yz$ rectangulum ſub partibus $x$ tranſcen-

* Vide *Schooteni Comment. in opera Cartes*, vol. 1, p. 345.

† Prodiit anno ſuperiore *Paduæ* liber cujus titulus ſaltem, deſideratum hoc tandem ſuppleri poſſe expectationem præbet, *Della poſſibilita della reale ſolutione analitica del caſo irreducibile*, auct. *Giambat. Nicolai*, Algeb. Profeſſ. At nondum ad manus pervenit.

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dere nequit quadratum ſemiſſis $x$, ſeu $\frac{1}{4}x^2$. § 46. Et proinde conditio $yz = -\frac{1}{3}q$ plane impoſſibilis eſt ubi $-\frac{1}{3}q$ ſit majus quam $\frac{1}{4}x^2$ ſeu (quod eodem redit) $\frac{1}{3}p^2$, adeoque $-\frac{q^3}{27}$ majus quam $\frac{1}{4}r^2$. per § 207. Et ex hypotheſi impoſſibili concluſionem impoſſibilem colligi mirum eſſe non debet.

212. Pergamus jam hujuſce Regulæ uſum exemplis illuſtrare.

*Ex.* 1. Sit æquatio reſolvenda $x^3 + 30x = 117$. Erit $\frac{1}{2}r = \frac{117}{2}$, $\frac{1}{3}q = 10$, $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}} = \sqrt{\frac{13689}{4} + 1000} = \sqrt{\frac{17689}{4}}$

$= \frac{133}{2}$, $y = \sqrt[3]{\frac{117}{2} + \frac{133}{2}} = \sqrt[3]{\frac{250}{2}} = \sqrt[3]{125} = 5$, & $z =$

$\sqrt[3]{\frac{117}{2} - \frac{133}{2}} = \sqrt[3]{\frac{-16}{2}} = \sqrt[3]{-8} = -2$. Ergo $y + z = 5$

$-2 = 3$, et reliquæ radices $-\frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} =$

$= \frac{-3 \pm 7\sqrt{-3}}{2}$.

*Ex.* 2. Proponatur æquatio $u^3 + 3u^2 + 9u = 13$. Deleto termino ſecundo, ſubſtituendo nimirum $x - 1$ pro $u$, erit $x^3 + 6x = 20$. Adeoque $\frac{1}{2}r = 10$, $\frac{1}{3}q = 2$, $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$

$= \sqrt{100 + 8} = 10{,}3923$. Ergo, $y + z = \sqrt[3]{: 10 + 10{,}3923}$

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$+ \sqrt[3]{10 - 10,3923} = 2,732 - 0,732 = 2.$ Et $-\frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} = -1 \pm 1,732\sqrt{-3}.$ Hinc $u = (x - 1 =)$ 1, vel $-2 \pm 1,732\sqrt{-3}.$

*Ex.* 3. Sit æquatio $u^6 - 3u^4 - 2u^2 = 8.$ Depreſsâ æquatione et ſublato termino ſecundo, ſubſtituendo utique $x + 1$ pro $u^2$, prodit $x^3 - 5x = 12.$ Adeoque $\frac{1}{2}r = 6$, $\frac{1}{3}q = \frac{5}{3}$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{36 - \frac{125}{27}} = 5,6009.$ Ergo $y + z = \sqrt[3]{6 + 5,6009} + \sqrt[3]{6 - 5,6009} = 2,26376 + ,73624 = 3.$ Et proinde $u^2 = (x + 1 =)\ 4.$ Reliquæ radices ut in præcedentibus exemplis prodeunt impoſſibiles.

213. Aliquando binomiæ quantitatis (qualem parit hæc æquationis cubicæ reſolutio) ex una parte rationali (ſive hæc integra fuerit ſive fracta) et alterâ irrationali, vel ex duabus irrationalibus ad invicem incommenſurabilibus conflatæ, radix cubica per binomiam ſimpliciorem deſignari poteſt; ut olim expoſuit *Bombellius*. Ex methodis autem in hunc finem excogitatis, *Walliſianam* * licet tentando ſit, in praxi commodiſſimam reperi. Hanc idcirco in compendium redactam promotamque jam tradam.

* Vide *Wallis*, *Opera Mathemat.* vol. 2, p. 188, & ſeq.

## INVESTIGATIO RADICIBUS BINOMII CUBI.

### I.

214. Sit quantitas $A \pm \sqrt{B^2Q}$ ex unâ parte integrâ A, et alterâ irrationali $\sqrt{B^2Q}$ conflata. Reducatur $\sqrt{B^2Q}$ ad formam ſimpliciſſimam $B\sqrt{Q}$ extrahendo quicquid eſt rationale, ſi modo B deſignet numerum ab unitate diverſum. Sit radix quæſita $x \pm b\sqrt{y}$, et manifeſtum eſt, quod $b\sqrt{y}$ debet eſſe pars aliquota $B\sqrt{Q}$, adeoque $b$ pars aliquota B et $\sqrt{y}$ ipſi $\sqrt{Q}$ æqualis; niſi enim hæc ſurda $\sqrt{Q}$ ingrederetur radicem ingredi non poſſet cubum. Erit igitur $x^3 \pm 3x^2b\sqrt{y} + 3xb^2y \pm b^3y\sqrt{y} = (\overline{x \pm b\sqrt{y}}^3 =)\ A \pm B\sqrt{Q}$. Adeoque pars integra cubi $x^3 + 3xb^2y = A$, & pars irrationalis $3x^2b\sqrt{y} + b^3y\sqrt{y}$ ſeu $(3x^2b + b^3y)\sqrt{y} = B\sqrt{Q}$. Unde dividendo per $\sqrt{y} = \sqrt{Q}$, prodit $3x^2b + b^3y = B$, adeoque per reductionem, $x^2 = \frac{B}{3b} - \frac{b^2y}{3}$, et $x = \pm \sqrt{\frac{B}{3b} - \frac{b^2y}{3}}$. Sigillatim jam ſubſtitue pro $b$, integros 1, 2, 3, &c. aut fractos $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ &c. donec $\sqrt{\frac{B}{3b} - \frac{b^2y}{3}}$ emergat rationalis. Deinde ſubſtituto valore rationali $x$, ſi $x^3 + 3xb^2y$ evadit æqualis A, fiet quæſitum, et innoteſcet utraque pars radicis $x$ & $b\sqrt{y}$, ſi modo radix extrahi poteſt.

Integros 1, 2, 3, &c. aut integrorum ſemiſſes $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c. ſubſtituendos eſſe pro $b$, quò partibus radicis in ſe ductis emergat A integer, ita demonſtrare licet.

Sit radix quæſita $\frac{x}{m} + \frac{b\sqrt{y}}{n}$, deſignantibus $m$ et $n$ integros ad $x$ & $b\sqrt{y}$ reſpective primos; eritque $\frac{x^3}{m^3} + \frac{3x^2b\sqrt{y}}{m^2n} + \frac{3xb^2y}{mn^2} + \frac{b^3y\sqrt{y}}{n^3}$ radicis cubus, ubi $\frac{x^3}{m^3} + \frac{3xb^2y}{mn^2}$ ſeu A, necnon $\frac{3x^2b}{m^2n} + \frac{b^3y}{n^3}$ ſeu B, debet eſſe integer. Ponatur $\frac{3x^2b}{m^2n} + \frac{b^3y}{n^3} = c$ int. Et concinnando erit $y = \frac{n^3c}{b^3} - \frac{3n^2x^2}{m^2b^2}$ integer, adeoque $\frac{3n^2}{m^2b^2}$ $x^2$ integer, adeoque coefficiens $\frac{3n^2}{m^2b^2}$ integer, et $n$ ex multiplis $mb$. Pone igitur $n = mb \times d$. Et ſubſtituto hoc valore $n$, prodit $y = m^3cd^3 - 3x^2d^2$. Atque ſubſtitutis iterum valoribus $n$ & $y$ jam inventis, prodit $\frac{x^3}{m^3} + \frac{3xb^2y}{mn^2} = 3xcd - \frac{8x^3}{m^3}$ integer, adeoque $\frac{8x^3}{m^3}$ integer, adeoque coefficiens $\frac{8}{m^3}$ integer. Et proinde $m^3 = 1$ vel 8, et $m = 1$ vel 2. Cæterum fuit $y = d^2 \times \overline{m^3cd - 3x^2}$, adeoque $\sqrt{y} = d\sqrt{m^3cd - 3x^2}$, adeoque altera pars radicis $\frac{b\sqrt{y}}{n} = \frac{bd\sqrt{m^3cd - 3x^2}}{mbd} = \frac{\sqrt{m^3cd - 3x^2}}{m}$. Ergo, utraque pars radicis communem habet denominatorem $m = 1$ ubi $x$ & $b$ ſunt integri, vel $= 2$ ubi ſunt fracti.

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215. Porro, ne operam fruſtrà inſumamus tentando ſucceſſive valores inutiles $b$, obſervandum eſt quod creſcente $b$, decreſcet $x = \sqrt{\frac{B}{3b} - \frac{b^2y}{3}}$; aut vice versâ. Et proinde ſi juſto majus aſſumatur $b$, juſto minus emerget $x$, adeoque $x^3 + 3b^2y$ minus quam A; aut vice versâ.

Secundum Methodum *Walliſianam* radicibus exquirendis ſolummodo adhibentur valores affirmativi $x$; at ope valorum negativorum $x$ aliæ inſuper radices explorari poſſunt.

216. *Ex.* 1. Si quæratur radix cubica quantitatis $10 + \sqrt{108}$ ſeu $10 + 6\sqrt{3}$; ſubſtituto 1 pro $b$, prodit $x = \sqrt{\frac{B}{3b} - \frac{b^2y}{3}}$ $= \sqrt{\frac{6}{3} - \frac{3}{3}} = 1$. Deinde ſubſtituto 1 pro $x$ in æquationem $x^3 + 3xb^2y = A$, prodit $1 + 9 = 10$, ut oportet. Ergo radix tentanda $1 + \sqrt{3}$. Hujus autem cubus per multiplicationem dat $10 + \sqrt{108}$; itaque fiet quæſitum.

*Ex.* 2. Sit radix cubica extrahenda ex $\sqrt{968} + 25$ ſeu $22\sqrt{2} + 25$. Subſtituto 1 pro $b$, prodit $x = \sqrt{\frac{22}{3} - \frac{2}{3}}$ $= \sqrt{\frac{20}{3}}$. Res igitur non ſuccedit, emergente $x$ irrationali. Subſtituatur iterum 2 pro $b$, et prodit $x = \sqrt{\frac{22}{6} - \frac{8}{3}} =$ $\sqrt{\frac{6}{6}} = 1$. Deinde, ſubſtituto 1 pro $x$, æquatio $x^3 + 3xb^2y$



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$= A$, evadit $1 + 24 = 25$, ut oportet. Ergo radix tentanda $2\sqrt{2} + 1$.

*Ex.* 3. Si radix cubica extrahenda ſit ex $2 + \sqrt{5}$, ſubſtituto 1 pro $b$ prodit $x = \sqrt{\frac{1}{3} - \frac{5}{3}} = \sqrt{-\frac{4}{3}}$ impoſſibilis, quod indicio eſt $b$ non ſortiri valorem integrum. Subſtituatur itaque $\frac{1}{2}$ pro $b$, et prodit $x = \sqrt{\frac{2}{3} - \frac{5}{12}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$, quo ſubſtituto, &c. evadit $\frac{1 + 15}{8} = 2$; ut oportet. Adeoque radix tentanda $\frac{1}{2} + \frac{1}{2}\sqrt{5}$. Atque hic quidem notandum, quod radicibus eruendis ubi $x$ & $b$ ſunt fracti, per ſe impares ſunt methodus *Newtoniana* & *Clairautiana*, atque idcirco ulteriore egent artificio quo hujuſmodi exemplis accommodentur.

217. Aliquando formam irrationalem induunt $x$ & $b$. Nam

*Ex.* 4. Si radix cubica extrahenda ſit ex $68 - \sqrt{4374}$ ſeu, $68 - 27\sqrt{6}$, valoribus $x$ & $b$ rationalibus fruſtra tentatis, radicem nihilominus eruere licet multiplicando quantitatem datam per $\frac{2}{2}$, unde prodit $\frac{136 - 54\sqrt{6}}{2}$, idem ſcilicet quantitatis valor. Dein extractâ ſeorſim numeratoris ac denominatoris radice cubicâ invenietur radix quæſita, $\frac{4 - \sqrt{6}}{\sqrt[3]{2}}$.

## II.

218. Si quantitatis $A \pm B\sqrt{Q}$ pars A ſit fractio, reducentur partes ad communem denominatorem. Dein ut antea, extractâ ſeorſim numeratoris ac denominatoris radice cubica orietur radix quæſita, ſi modo extrahi poteſt.

*Ex.* 5. Si ex $\sqrt{242} - 12\frac{1}{2}$ radix cubica extrahenda ſit; reductis partibus ad communem denominatorem fiet $\frac{\sqrt{968} - 25}{2}$. Dein extractâ radice cubica, &c. orietur $\frac{2\sqrt{2} - 1}{\sqrt[3]{2}}$ radix quæſita.

## III.

219. Quantitatis $A \pm B\sqrt{-Q}$ ſit una pars A rationalis, altera $B\sqrt{-Q}$ impoſſibilis. Deſignante $x \pm b\sqrt{-y}$ radicem, multiplicando erit $x^3 \pm 3x^2b\sqrt{-y} - 3xb^2y \mp b^3y\sqrt{-y} = A \pm B\sqrt{-Q}$. Adeoque $x^3 - 3xb^2y = A$, & $3x^2b - b^3y = B$, et $x = \pm\sqrt{\frac{B}{3b} + \frac{b^3y}{3}}$. Deinde operando ut antea, eruetur radix, ſi modo extrahi poteſt *.

* His veſtigiis inſiſtendo methodum proferre poſſumus extrahendi radicem quadraticam binomii $A \pm B\sqrt{Q}$, ex parte integra A & irrationali $B\sqrt{Q}$ conflati,

*Ex.* 6. Sit radix cubica extrahenda ex $-5 \pm \sqrt{-2}$. Ponatur $b = 1$, eritque $x = \sqrt{\frac{1}{3} + \frac{2}{3}} = 1$. Quo valore $x$ ſubſtituto prodit $x^3 - 3xb^2y = 1 - 6 = -5$, ut oportet. Ergo radix tentanda $1 \pm \sqrt{-2}$, quæ utique per multiplicationem dat $-5 \pm \sqrt{-2}$.

220. Miror itaque unde *Saunderſon* ad *De Moivre* ſcribens, profitetur, "Se nihil ad rem deprehendiſſe, de ex-" trahenda radice cubica binomii impoſſibilis, inſtar "$-5 + \sqrt{-2}$ vel $-5 - \sqrt{-2}$, ne quidem apud ipſum "*Wallis* qui hoc aggreditur." Magiſque miror unde in

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*Newtonianâ* § 52, in praxi commodiorem. Deſignante enim $x \pm b\sqrt{y}$ radicem, erit $x^2 \pm 2xb\sqrt{y} + b^2y = A \pm B\sqrt{Q}$, adeoque $x^2 + b^2y = A$; $2xb = B$, et inde $x = \frac{B}{2b}$. Deinde ſigillatim ſumendo $b = 1, 2, 3$, &c. ſubſtituantur valores $x$ reſultantes, donec $x^2 + b^2y$ evadat æqualis A; et fiet quæſitum, ſi modo radix extrahi poteſt. Ut ſi extrahenda ſit radix ex $3 + \sqrt{8}$ ſeu $3 + 2\sqrt{2}$; ſume $b = 1$, eritque $x = \frac{2}{2} = 1$, quo ſubſtituto prodit $x^2 + b^2y = 1 + 2 = 3$, ut oportet. Ergo radix tentanda $1 + \sqrt{2}$.

Quòd ſi $B\sqrt{-Q}$ ſit impoſſibilis, erit $x^2 - b^2y = A$, cæteris manentibus. (Radicem, in hoc caſu, deſignante $x \pm b\sqrt{-y}$.) Ut ſi quæratur radix quadratica quantitatis impoſſibilis $1 \pm 4\sqrt{-3}$; ſume $b = 1$, eritque $x = \frac{4}{2} = 2$. Quo ſubſtituto, prodit $x^2 - b^2y = 4 - 3 = 1$, ut oportet. Ergo radix tentanda $2 \pm \sqrt{-3}$.

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regulam *Walliſianam* animadvertit. *De Moivre*, quaſi "*peti-* "*tio principii* foret, aut ſaltem tentamen quod radice eruen- "dâ ſæpe deficeretur." In Logicis certe parum verſatum fuiſſe videtur *De Moivre*, aut hanc regulam ad *petitionem principii* vix retulerit; et æquo ſaltem jure, in regulam *pro inventione Diviſorum*, adeo decantatam, animadvertere potuerit, cum hæc quoque non niſi tentando ſuccedit. Vide *Append. ad Saunderſon, Algeb. vol.* II. *p.* 743.

*Ex.* 7. Si quæratur $\sqrt[3]{:}\ 81 + \sqrt{-2700} = \sqrt[3]{:}\ 81 + 30\sqrt{-3}$; ponendo $b = 1$, erit $x = \sqrt{11}$. Res igitur non ſuccedit, emergente $x$ irrationali. Rurſus, ponendo $b = 2$, erit $x = \pm\sqrt{9} = \pm 3$. Si ſubſtituatur $+3$ pro $x$, prodit $x^3 - 3xb^2y = 27 - 108 = -81$, adeoque res non ſuccedit, quoniam ſigno affirmativo affici debet 81. At ſi ſubſtituatur $-3$ pro $x$, prodit $-27 + 108 = +81$; ut oportet. Ergo radix tentanda $-3 + 2\sqrt{-2}$.

Altera quoque radix erui poteſt ſumendo $b = \frac{1}{2}$, unde erit $x = \sqrt{20 + \frac{1}{4}} = \frac{9}{2}$, adeoque $x^3 - 3xb^2y = \frac{729}{8} - \frac{81}{8} = 81$; ut oportet. Ergo radix tentanda $\frac{9}{2} + \frac{1}{2}\sqrt{-3}$.

221. Cæterum in hujuſmodi operationibus ſi cubi partes communem habent diviſorem cubicum, ſatius erit radices factorum ſeorſim extrahere. Sic in noviſſimo exemplo, $81 + \sqrt{-2700} = 27 \times \overline{3 + \sqrt{-\frac{100}{27}}}$, radicibus factorum

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ſeorſim extractis, valet radix quæſita $3 \times \overline{-1 + 2\sqrt{-\frac{1}{3}}}$ $= -3 + 2\sqrt{-\frac{9}{3}} = -3 + 2\sqrt{-3}$, ut in priore caſu; et altera radix $3 \times \overline{\frac{3}{2} + \frac{1}{2}\sqrt{-\frac{1}{3}}} = \frac{9}{2} + \frac{1}{2}\sqrt{-3}$, ut in poſteriore.

## IV.

222. Sit quantitas $A\sqrt{P} \pm B\sqrt{Q}$, cujus utraque pars irrationalis eſt. Deſignante $a\sqrt{x} \pm b\sqrt{y}$ radicem, multiplicando erit $a^3x\sqrt{x} \pm 3a^2xb\sqrt{y} + 3ab^2y\sqrt{x} \pm b^3y\sqrt{y}$ $= A\sqrt{P} \pm B\sqrt{Q}$; adeoque, $a^3x + 3ab^2y = A$ & $3a^2xb + b^3y = B$, unde prodit $a = \sqrt{\frac{B}{3xb} - \frac{b^2y}{3x}}$. Pro $x$ ſubſtitue jam valorem ſuum P; et ſubſtitue ſigillatim pro $b$ integros 1, 2, 3, &c. donec eruatur valor $a$ rationalis; quo ſubſtituto, ſi $a^3x + 3ab^2y$ prodeat æqualis A, fiet quæſitum.

*Ex.* 8. Si radix cubica extrahenda eſt ex $\sqrt{5408} + \sqrt{5400}$ ſeu $52\sqrt{2} \pm 30\sqrt{6}$; ſume $b = 1$, evadetque $a = \sqrt{5 - 1}$ $= 2$. Et hoc valore $a$ ſubſtituto, prodit $a^3x + 3ab^2y = 16 + 36 = 52$, ut oportet. Ergo radix tentanda $2\sqrt{2} \pm \sqrt{6}$.

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## De resolutione æquationum cubicarum per extractionem radicis binomii cubi.

223. Aliquando radices in *caſu irreducibili* exprimere poſſibile erit, ubi ſcilicet radices valorum $y$ & $z$ ope regulæ jam allatæ cubice extrahere licet. Nam quod ſingulis membris videtur impoſſibile, mutua connexione deſtruetur *.

*Ex.* 1. Proponatur æquatio $x^3 - 15x = 4$. Erit $\frac{1}{2}r = 2$, $\frac{1}{3}q = 5$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{4 - 125} = \sqrt{-121} = 11\sqrt{-1}$,

* *Symbola imaginaria* proprie quidem calculum reſpuunt, cum ſint tantum impoſſibilitatis ſigna; veruntamen ubi quantitates reales deſignare ponuntur (ut in hoc caſu) quod mirum & paradoxi inſtar videatur, ex earum ſummis, productis & rationibus ſecundum regulas arithmeticas computatis, veræ & legitimæ ſequuntur concluſiones. Ita ut Geometria, etſi calculum quantitatum impoſſibilium proprie non agnoſcit, tamen concluſiones inde elicitas demonſtrationibus accuratiſſimis munit atque ſtabilit. Cæterum ſignis imaginariis quantitates adumbrari reales ibi tantum conceditur, ubi quæſtio ſit de menſuris angulorum aut rationum, idque propter arctum nexum & affinitatem inter arcus circulares & ſectores hyperbolicos intercedentem. Atque hinc oritur quod concluſiones his propriæ ad illos per analogiam tranferuntur. Sic iidem prodeunt valores rectangulorum ſub ſinubus & coſinubus ſectorum circulorum, atque ſub ordinatis & abſciſſis ſectorum hyperbolicorum; calculo in utroque caſu ſimiliter inſtituto, niſi quod in priore, mediantibus ſignis imaginariis, in poſteriore, ſignificantibus ad eandem concluſionem devenitur. At hoc tantum hic loci monuiſſe fas eſt. Adeat lector horum ſtudioſus, *Playfair, on the arithmetic of impoſſible quantities, Phil. Tranſact.* an. 1778, p. 318.

adeoque $y + z = \sqrt[3]{:2 + 11\sqrt{-1}} + \sqrt[3]{:2 - 11\sqrt{-1}}$ $= \overline{2 + \sqrt{-1}} + \overline{2 - \sqrt{-1}} = 4$. Itemque $-\frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} = -2 \pm \sqrt{-1}\,.\,\sqrt{-3} = -2 \pm \sqrt{3}$. In hoc exemplo igitur prima radix 4 eſt maxima.

*Ex.* 2. Sit $x^3 - 6x = 4$. Erit $\frac{1}{2}r = 2$, $\frac{1}{3}q = 2$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{4 - 8} = \sqrt{-4} = 2\sqrt{-1}$. Adeoque $y + z$ $= \sqrt[3]{:2 + 2\sqrt{-1}} + \sqrt[3]{:2 - 2\sqrt{-1}} = -1 + \sqrt{-1}$ $-1 - \sqrt{-1} = -2$. Itemque $-\frac{y+z}{2} + \frac{y-z}{2}$ $\sqrt{-3} = 1 \pm \sqrt{-1}\,.\,\sqrt{-3} = 1 \pm \sqrt{3}$, ubi prima radix $-2$ eſt media.

*Ex.* 3. Sit $x^3 - 9x = -10$. Erit $\frac{1}{2}r = 5$, $\frac{1}{3}q = 3$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{25 - 27} = \sqrt{-2}$. Adeoque $y + z =$ $\sqrt[3]{:-5 + \sqrt{-2}} + \sqrt[3]{:-5 - \sqrt{-2}} = 1 + \sqrt{-2} + 1 -$ $\sqrt{-2} = 2$. Itemque $-\frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} = -1$ $\pm \sqrt{-2}\,.\,\sqrt{-3} = -1 \pm \sqrt{6}$. Ubi prima radix 2 eſt media.

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*Ex.* 4. Sit $x^3 - 15x = 22$. Erit $\frac{1}{2}r = 11$, $\frac{1}{3}q = 5$, $\sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \sqrt{121-125} = \sqrt{-4}$. Adeoque $y + z = \sqrt[3]{11 + \sqrt{-4}} + \sqrt[3]{11 - \sqrt{-4}} = -1 + \sqrt{-4} - 1 - \sqrt{-4} = -2$. Itemque $-\frac{y+z}{2} \pm \frac{y-z}{2}\sqrt{-3} = 1 \pm \sqrt{-4} \cdot \sqrt{-3} = 1 \pm \sqrt{12}$. Ubi prima radix $-2$ eſt minima.

Hinc liquet, primam radicem hac methodo inventam trium radicum realium unamquamque indifferenter deſignare poſſe, non maximam ſolùm, ut vulgo perhibent.

In *caſu reducibili* quoque aliquando facilius erui poſſunt radices ope regulæ allatæ. Ut ſi repetatur æquatio $x^3 + 6x = 20$, prodit $y + z = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = 1 + \sqrt{3} + 1 - \sqrt{3} = 2$; ut antea, § 212.

224. Cæterum deficiente *Cardani* regula ubi binomiarum impoſſibilium inde reſultantium radices cubicæ accurate extrahi nequeunt, ad theorema binomium confugiendum erit, cujus ope harum radices *ſemper* per ſeries infinitas proxime exponi poſſunt, ſecundum methodum quam primus excogitaſſe videtur *Leibnitz*, § 205, tradiderunt verò *Ronayne*, *Algebra*, 8vo Lond. 1717, & Edit. 2da 1727, et *Nicole*. *Mem. de l'Acad. Paris*, 1738.

## De resolutione æquationum cubicarum per series infinitas.

225. Abbreviandi gratiâ ponatur $\frac{1}{2}r = a$, & $\sqrt{\frac{r^2}{4} \pm \frac{q^3}{27}} = \sqrt{\pm c}$, ſcilicet $= \sqrt{c}$ in caſu reducibili, et $\sqrt{-c}$ in caſu irreducibili. Per. theorema binomium, erit in priore caſû,

I.

$$y = (a + \sqrt{c})^{\frac{1}{3}} = \sqrt[3]{a} \times : 1 + \frac{\sqrt{c}}{3a} - \frac{2c}{3.6a^2} + \frac{2.5c\sqrt{c}}{3.6.9a^3} - \frac{2.5.8c^2}{3.6.9.12a^4}$$

$$z = (a - \sqrt{c})^{\frac{1}{3}} = \sqrt[3]{a} \times : 1 - \frac{\sqrt{c}}{3a} - \frac{2c}{3.6a^2} - \frac{2.5c\sqrt{c}}{3.6.9a^3} - \frac{2.5.8c^2}{3.6.9.12a^4}$$

$$\therefore\ y + z = 2\sqrt[3]{a} \times : 1 \quad * \quad - \frac{2c}{3.6a^2} \quad * \quad - \frac{2.5.8c^2}{3.6.9.12a^4} \text{ \&c.}$$

Exponet itaque ſeries $2\sqrt[3]{a} \times : 1 - \frac{2c}{3.6a^2} -$ &c. primam radicem, qualem invenerunt *Ronayne* & *Nicole*. Erit quoque

$$y - z = 2\sqrt[3]{a} \times : \quad * + \frac{\sqrt{c}}{3a} \quad * \quad + \frac{2.5c\sqrt{c}}{3.6.9a^3} \quad * \text{ \&c.}$$

$$= 2\sqrt[3]{a} \times \frac{\sqrt{c}}{a} \times : \frac{1}{3} \quad * \quad + \frac{2.5c}{3.6.9a^2} \quad * \text{ \&c.}$$

$$= \frac{2\sqrt[3]{a} \times \sqrt{c}}{\sqrt[3]{a^3}} \text{ ſeu } \frac{2\sqrt{c}}{\sqrt[3]{a^2}} \times : \frac{1}{3} + \frac{2.5c}{3.6.9a^2} + \frac{2.5.11c^2}{3.6.9.12.15a^4} \text{ \&c.}$$

Adeoque reliquæ radices, quales expoſuit *Clairaut*,

$$\left.\begin{array}{l} -\frac{y+z}{2} \\ \pm\frac{y-z}{2}\sqrt{-3} \end{array}\right\} = \left\{\begin{array}{l} -\sqrt[3]{a} \times : 1 - \frac{2c}{3.6a^2} - \frac{2.5.8c^2}{3.6.9.12a^4} \text{ \&c.} \\ \pm\frac{\sqrt{-3c}}{\sqrt[3]{a^2}} \times : \frac{1}{3} + \frac{2.5c}{3.6.9a^2} + \frac{2.5.8.11c^2}{3.6.9.12.15a^4} \text{ \&c.} \end{array}\right.$$

II.

Et in poſteriore caſu, erunt radices,

$$y + z = 2\sqrt[3]{a} \times : 1 + \frac{2c}{3.6a^2} - \frac{2.5.8c^2}{3.6.9.12a^4} + \text{\&c.}$$

$$\text{et } \left.\begin{array}{l} -\frac{y+z}{2} \\ \pm\frac{y-z}{2}\sqrt{-3} \end{array}\right\} = \left\{\begin{array}{l} -\sqrt[3]{a} \times : 1 + \frac{2c}{3.6a^2} - \frac{2.5.8c^2}{3.6.9.12a^4} + \text{\&c.} \\ \pm\frac{\sqrt{3c}}{\sqrt[3]{a^2}} \times : \frac{1}{3} - \frac{2.5c}{3.6.9a^2} + \frac{2.5.8.11c^2}{3.6.9.12.15a^4} - \text{\&c.} \end{array}\right.$$

Mutatis ſcilicet ſignis terminorum alternorum a ſecundo incipientium, utpote qui poteſtates impares quantitatis jam negativæ $-c$ involvunt.

226. Si $-\frac{q^3}{27}$ ſit duplum $\frac{1}{4}r^2$, ſimplicius efferri poſſunt hæ ſeries. Nam ob $\sqrt{c}$ in hoc caſu æquale $a$, erit $\frac{\sqrt{c}}{a} = 1$. Adeoque, prima radix,

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$$y + z = 2\sqrt[3]{a} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \frac{2.5.8.11.14}{3.6.9.12.15.18} - \&c.$$

$$\text{Et reliquæ,} = \begin{cases} -\sqrt[3]{a} \times : 1 + \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} + \&c. \\ \pm \sqrt[3]{a} \, . \, \sqrt{-3} \times : \frac{1}{3} - \frac{2.5}{3.6.9} + \frac{2.5.8.11}{3.6.9.12.15} - \&c. \end{cases}$$

227. Cæterum hæ ſeries radicibus eruendis *directe* inſerviunt tantùm ubi convergunt, hoc eſt, ſi $-\frac{q^3}{27}$ non ſit duplo majus quam $\frac{1}{4}r^2$, adeoque $\sqrt{c}$ non majus quam $c$. Et quo proprius $-\frac{q^3}{27}$ accedit ad $\frac{1}{4}r^2$, eo minus evadet $\sqrt{c}$ reſpectu $a$, et proinde eo citius converget ſeries ex evolutione $(a \pm \sqrt{c})^{\frac{1}{3}}$ oriunda. Quod ſi $\frac{q^3}{27}$ affirmativum ſit, aut ſi negativum ſit & ſimul duplo majus quam $\frac{1}{4}r^2$, foret $\sqrt{c}$ majus quam $a$, unde divergeret ſeries. In hoc caſu igitur *invertendi* ſunt termini $a$ & $\sqrt{c}$; nam ex evolutione $(\sqrt{c} + a)^{\frac{1}{3}}$ prodibit ſeries convergens ejuſdem valoris. Adeoque in caſu reducibili erit

III.

$$y = \sqrt[3]{:\overline{\sqrt{c} + a}}, \& \; z = \sqrt[3]{:\overline{-\sqrt{c} + a}} = -\sqrt[3]{:\overline{\sqrt{c} - a}}.$$

$$\therefore y = (\sqrt{c} + a)^{\frac{1}{3}} = \sqrt[3]{c^{\frac{1}{2}}} \times : 1 + \frac{a}{3\sqrt{c}} - \frac{2a^2}{3.6c} + \frac{2.5a^3}{3.6.9c\sqrt{c}} - \&c.$$

$$z = -(\sqrt{c} - a)^{\frac{1}{3}} = \sqrt[3]{c^{\frac{1}{2}}} \times : -1 + \frac{a}{3\sqrt{c}} + \frac{2a^2}{3.6c} + \frac{2.5a^3}{3.6.9c\sqrt{c}} + \&c.$$

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Et $y + z = 2\sqrt[3]{c^{\frac{1}{2}}} \times : * + \frac{a}{3\sqrt{c}} \quad * + \frac{2 \cdot 5a^3}{3.6.9c\sqrt{c}} \quad * \quad + \text{\&c.}$

$$= 2\sqrt[3]{c^{\frac{1}{2}}} \cdot \frac{a}{\sqrt{c}} \text{ ſeu } \frac{2a}{\sqrt[6]{c^2}} \text{ ſeu } \frac{2a}{\sqrt[3]{c}} \times : \frac{1}{3} + \frac{2.5a^2}{3.6.9c} + \frac{2.5.8.11a^4}{3.6.9.12.15c^2} + \text{\&c.}$$

exponet primam radicem, qualem invenit *Clairaut*.

Itemque,

$$\left.\begin{array}{l} -\frac{y+z}{2} \\ \pm\frac{y-z}{2}\sqrt{-3} \end{array}\right\} = \left\{\begin{array}{l} -\frac{a}{\sqrt[3]{c}} \times : -\frac{1}{3} + \frac{2.5a^2}{3.6.9c} + \frac{2.5.8.11a^4}{3.6.9.12.15c^2} + \text{\&c.} \\ \pm\sqrt[3]{c^{\frac{1}{2}}} \cdot \sqrt{-3} \times : 1 - \frac{2a^2}{3.6c} - \frac{2.5.8a^4}{3.6.9.12c^2} - \text{\&c.} \end{array}\right.$$

reliquas, quales invenit *Hutton*.

## IV.

Et in caſu irreducibili erit prima radix

$$= \frac{-2a}{\sqrt[3]{c}} \times : \frac{1}{3} - \frac{2 \cdot 5a^2}{3 \cdot 6 \cdot 9c} + \frac{2 \cdot 5 \cdot 8 \cdot 11a^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c} - \text{\&c.}$$

Et reliquæ,

$$= \left\{\begin{array}{l} \frac{a}{\sqrt[3]{c}} \times : \frac{1}{3} - \frac{2 \cdot 5a^2}{3 \cdot 6 \cdot 9c} + \text{\&c.} \\ \pm\sqrt[3]{c^{\frac{1}{2}}} \cdot \sqrt{3} \times : 1 + \frac{2a^2}{3 \cdot 6c} - \frac{2 \cdot 5 \cdot 8a^4}{3 \cdot 6 \cdot 9 \cdot 12c^2} + \text{\&c.} \end{array}\right.$$

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228. Pergamus jam harum ſerierum uſum exemplis illuſtrare.

*Ex.* 1. Sit æquatio $x^3 - 12{,}03x = 16$. Erit $a = 8$, $-c = -{,}481201$ adeoque ob $c$ negativum & multò minus quam $a^2$, ad hibendæ ſeries II. citiſſimè convergentes. Erit igitur

$$\frac{c}{a^2} = \frac{,481201}{64} = ,007518 \; . \; \frac{c^2}{a^4} = ,0000565$$

$$\frac{1c}{9a^2} = \frac{,007518}{9} = ,000835 \; . \; \frac{10c^2}{243a^4} = \frac{,000565}{243} = ,0000023$$

$$\begin{array}{r} 1{,}000835 \\ -\;,000002 \\ \hline \end{array} \qquad \frac{\sqrt{3c}}{\sqrt[3]{a^2}} = \frac{,12015}{4}$$

$$1 + \frac{c}{9a^2} - \frac{10c^2}{243a^4} = 1{,}000833$$

$$\begin{array}{r} \times\; 2\sqrt[3]{a} = 4 \\ \hline \end{array} \qquad \frac{\sqrt{3c}}{\sqrt[3]{a^2}} \times \frac{1}{3} = \frac{,03003}{3} = ,010013.$$

Prima radix, $\quad 4{,}003332$

$$-\sqrt[3]{a} \times : 1 + \frac{c}{9a^2} \quad = -\,2{,}001666$$

$$\pm \frac{\sqrt{3c}}{\sqrt[3]{a^2}} \times \frac{1}{3} \quad - \; - \quad = \pm \;\; ,010013$$

Media radix, - - $-\,2{,}011679$

Ultima, - - - - - $-\,1{,}991653$.

*Ex.* 2. Sit æquatio $x^3 - 50x = 120$. Erit $a = 60$, $c = -\,1029{,}629629$. Adeoque adhibendæ ſeries II. Erit igitur,

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$$\frac{c}{a^2} = ,286008 \; . \; \frac{c^2}{a^4} = ,081800 \; . \; \frac{c^3}{a^6} = ,023395$$

$$\frac{1c}{9a^2} = ,031778 \; . \; \frac{10c^2}{243a^4} = ,003353 \; . \; \frac{154c^3}{6561a^6} = ,000549$$

1,031778
+ 000549

1,032327
— ,003353

1,028974
× 2

2,057948
$\times \sqrt[3]{a} = 3,914867$

Prim. rad. $= 8,056592$ *

$\left\{ \begin{array}{r} -\ 4,028296 \\ \pm\ \ ,096647 \end{array} \right.$

Media, — 4,124943
Ultima, — 3,931649.

$$\frac{\sqrt{c}}{\sqrt[3]{a^2}} = \frac{,555777}{15,3263} = ,036262$$

$$\frac{10c}{162a^2} = \frac{2,86008}{162} = ,017654$$

$$\frac{1}{3} - \frac{10c}{162a^2} = +\ ,333333$$
— ,017654

+ ,315679
$\times \frac{\sqrt{c}}{\sqrt[3]{a^2}} = ,036262$

,096647

* Si ex diametro tribuſque circuli chordis contiguis factis fiat trapezium circulo inſcriptum, datis chordis $a$, $b$, $c$ erui poteſt diameter $x$ per reſolutionem æquationis $x^3 - (a^2 + b^2 + c^2)\, x = 2abc$, quam ex mutuis harum relationibus vario artificio educit *Newtonus*, *Arith. Univerſ.* Ed. 2, p. 101, &c. Ut ſi chordas reſpective deſignent numeri 3, 4, 5, reſolutio æquationis reſultantis $x^3 - 50x = 120$, dabit $x = 8,056$ &c. reliquæ verò radices — 4,124 &c. et — 3,931 &c. utpote negativæ aliò referuntur, nec ſi evaderent affirmativæ diametrum deſignare poſſent, ob limitationem in ſchemate, cum rectarum circulo inſcriptarum maximus prodit diameter. § 134. Vide *Maſeres*, *Diſſert. on Cardan's rule*, *Phil. Tranſ.* an. 1778.

229. Cæterum ubi plures ſerierum termini radicibus deſignandis requiruntur, in numeros profuſiores excurret calculus, plurimum vero minuetur labor combinando ſerierum terminos inter operandum inventos, & adhibendo logarithmos; ut in exemplis ſequentibus.

*Ex.* 3. Sit æquatio $x^3 - 15x = 22$. Erit $a = 11$ & $-c = -4$. Adeoque adhibendæ ſeries II. Erit quoque $\frac{c}{a^2} = \frac{4}{121} = \frac{1}{30,25}$. Et combinando terminos inter operandum inventos, erit

$$A = \qquad = 1,0000000$$

$$B = \frac{2c}{3.6a^2} A = \qquad 36731 \qquad C = \frac{5.8c}{9.12a^2} = -,0000450.$$

$$D = \frac{11.14c}{15.18a^2} C = \qquad 8$$

$$+\ 1,0036739$$
$$-\ 0,0000450$$

Prima ſeries $= 1,0066289$
$\times\ 2$

| | | | |
|---|---|---|---|
| | 2,0072578 | - - - log. | 0,3026031 |
| | $\times \sqrt[3]{11}$ | - - - - - | 0,3471309 |
| Prima radix | $= 4,4641016$ | - - - - | 0,6497340. |

Itemque,

$\alpha = \frac{1}{3} A = ,3333333$ | $\beta = \frac{5}{9} B = ,0020406$

$\gamma = \frac{11}{15} C = \quad 330$ | $\delta = \frac{17}{21} D = \quad 7$

$+ ,3333663$      $- ,0020413$

$- ,0020413$

$,3313250$

$\times 2$

| | | |
|---|---|---|
| $,6626500$ | - - | log. $\bar{1},8212842$ |
| $\times \sqrt{3}$ | - - - | $0,2385606$ |
| $\times \sqrt[3]{121}$ | - - - - | $0,6942618$ |
| Poſtrema ſeries $\pm ,2320508$ | - - - - | $\bar{1},3655830$ |

$- \frac{1}{2}$ pri. rad. $- 2,2320508$

Media rad. $= - 2,4641016$

Ult. radix $= - 2,0000000$

*Ex.* 4. In æquatione $x^3 - 9x = -10$, prodit $a = -5$, $c = 25 - 27 = -2$. Adhibendæ igitur ſeries II. Et exiſtente $\frac{c}{a^2} = \frac{2}{25} = \frac{8}{100} = ,08$. erit

| | |
|---|---|
| $A = \quad = 1$ | |
| $B = \frac{2c}{3.6a^2} A = 0{,}0088889$ | $C = \frac{5.8c}{9.12a^2} B = {,}0002634$ |
| $D = \frac{11.14c}{15.18a^2} C = \quad 120$ | $E = \frac{17.20c}{21.24a^2} D = \quad 7$ |
| $+ 1{,}0089009$ | $- {,}0002641$ |

$+ 1{,}0089009$
$- 0{,}0002641$
$+ 1{,}0086368$
$2$
$2{,}0172736$ - - - log. $0{,}3047649$
$\sqrt[3]{a} = -\sqrt[3]{5}$ - - - - $0{,}2329900$

Prima rad. $- 3{,}44948974$ - - - - $0{,}5377549$

Itemque,

$$\begin{array}{l|l} \alpha = \frac{1}{3} A = ,3333333 & \beta = \frac{5}{9} B = ,0049383 \\ \gamma = \frac{11}{15} C = \quad 1931 & \delta = \frac{17}{21} D = \quad 97 \\ \varepsilon = \frac{23}{27} E = \quad 6 & \overline{- ,0049480} \end{array}$$

$$\begin{array}{lrl} & \overline{+ ,3335270} & \\ & - ,0049480 & \\ & \overline{,3285790} & - \; - \; \text{log. } \bar{1},5166398 \\ & \frac{\sqrt{3c}}{\sqrt[3]{a^{2c}}} = \frac{\sqrt{6}}{\sqrt[3]{25}} & - \; - \; - \; - \; \bar{1},9230956 \\ \text{Altera ſeries} & \overline{\pm ,27525513} & \overline{\bar{1},4397354} \\ -\frac{1}{2} \text{ pri. rad.} & + 1,72474487 & - \\ & \overline{\phantom{2,00000000}} & \\ \text{Media radix} & 2,00000000 & \\ \text{Ultima radix} & 1,44948974. & \end{array}$$

*Ex.* 5. Sit æquatio $x^3 - 6x = 4$. Prodit $a = 2$, $\sqrt{c} = \sqrt{4 - 8} = \sqrt{-4}$; adeoque ob $\sqrt{c} = a$, adhibendæ ſeries § 226. His autem tardius convergentibus, tot proferre terminos quot radicibus proxime deſignandis poſtulantur moleſtum foret. Hujus radices § 223 ſupra, prodierunt $-2$, & $1 \pm \sqrt{3}$.

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*Ex.* 6. Sit æquatio $x^3 - 15x = 4$. Erit $a = 2$, $\sqrt{c} = \sqrt{4 - 125} = \sqrt{-121}$, adeoque ob $\sqrt{c}$ majus quam $a$, adhibendæ ſeries IV. Exiſtente igitur $\frac{a^2}{c} = \frac{4}{121}$. Erit,

| | + | | − |
|---|---|---|---|
| $A = \frac{1}{3}$ | $= ,3333333$ | $B = \frac{2.5a^2}{6.9c} A$ | $= ,0020406$ |
| $C = \frac{8.11a^2}{12.15c} B$ | $= 330$ | $D = \frac{14.17}{18.21} C$ | $= 7$ |
| | $+ ,3333663$ | | $- ,0020413$ |
| | $- ,0020413$ | | |

| | | |
|---|---|---|
| Prima ſeries | ,3313250 | log. $\bar{1},5202543$ |
| | $\frac{4}{\sqrt[3]{121}} = \sqrt[3]{\frac{64}{121}}$ | $\bar{2},9077982$ |
| Prima radix | $= ,2679492$ | $\bar{1},4280525$ |

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Itemque,

| | |
|---|---|
| $\alpha = \frac{3}{1}\, A = 1$ | |
| $\beta = \frac{9}{5}\, B = 0,0036731$ | $\gamma = \frac{15}{11}\, C = -,0000450$ |
| $\delta = \frac{21}{17}\, D = 8$ | |
| $+\ 1,0036739$ | |
| $-\ 0000450$ | |

| | |
|---|---|
| 1,0036289 - - | log. 0,0015732 |
| $\sqrt[3]{c\frac{1}{2}} = \sqrt[3]{11}$ - - - | 0,3471309 |
| $\sqrt{3}$ - - - - - - | 0,2385606 |
| $\pm\ 3,8660254$ - - - - | 0,5872647 |
| $-\frac{1}{2}$ primæ rad. $+\ 0,1339746$ | |
| Secunda radix $+\ 4,0000000$ | |
| Tertia radix $-\ 3,7320508.$ | |

230. Poſſint quoque ſeries utriuſque claſſis tam directæ quam inverſæ, in præcedentibus adhibitæ in alias quæ citius convergant tranſmutari. Verum de his diſſerere non patitur noſtri inſtituti ratio. Conſulat lector harum ſtudioſus auctorem ſupra laudatum, *Hutton*, qui rem totam penitius explorat, fuſiuſque explicat.

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## De Æquationum biquadraticarum resolutione.

229. Æquationum biquadraticarum reſolutionem primus invenit *Ferrarius*. Docuit vero *Carteſius* methodo inſequente radices æquationum biquadraticarum mediantibus cubicis eruere & exprimere.

Tollendus eſt primum ſecundus æquationis terminus. Sit æquatio reſultans $x^4 + Qx^2 + Rx + S = 0$. Pone hanc multiplicatione duarum quadraticarum $x^2 + ex + f = 0$, & $x^2 - ex + g = 0$ generari, id eſt, eandem eſſe cum hac $x^4 * + (f + g - e^2)\, x^2 + (eg - ef)\, x + fg = 0$, ubi neceſſe eſt ut quadraticarum termini ſecundi ſint iidem & contrariis ſignis affecti, quo evaneſcat ſecundus terminus biquadraticæ reſultantis. Et collatis biquadraticarum terminis homologis, fiet $f + g - e^2 = Q$, $eg - ef = R$, & $fg = S$. Quare erit

$$f + g = e^2 + Q$$
$$f - g = -\frac{R}{e}$$

$$\text{Adeoque } 2f * = e^2 + Q - \frac{R}{e}$$
$$\text{Et } - * 2g = e^2 + Q + \frac{R}{e}$$

$$\text{Adeoque } 4fg = e^4 + 2Qe^2 + Q^2 - \frac{R^2}{e^2} = 4S$$

$$\text{Sive } - - - e^6 + 2Qe^4 + (Q^2 - 4S)\, e^2 - R^2 = 0$$

$$\text{Et poſito } y = e^2,\ y^3 + 2Qy^2 + (Q^2 - 4S)\, y - R^2 = 0.$$

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Æquationis cubicæ jam reſultantis extrahatur radix. Dein habitâ illa radice regrediendum erit, ponendo $\sqrt{y} = e$, $\frac{e^2 + Q - \frac{R}{e}}{2} = f$, $\frac{e^2 + Q + \frac{R}{e}}{2} = g$, & æquationes duæ $x^2 + ex + f = 0$, & $x^2 - ex + g = 0$, extractis earum radicibus dabunt quatuor radices æquationis biquadraticæ $x^4 + Qx^2 + Rx + S = 0$.

Ut ſi proponatur æquatio $x^4 - x^3 - 5x^2 + 12x - 6 = 0$. Tolle ſecundum terminum ſcribendo $v + \frac{1}{4}$ pro $x$, & orietur $v^4 - \frac{43}{8}v^2 + \frac{75}{8}v - \frac{851}{256} = 0$. Ad tollendas fractiones ſcribe $\frac{1}{4}z$ pro $v$, & orietur $z^4 - 86z^2 + 600z - 851 = 0$. Hic eſt $-86 = Q$, $600 = R$, & $-851 = S$, adeoque $y^3 + 2Qy^2 + (Q^2 - 4S)y - R^2 = 0$, ſubſtitutis æquipollentibus fiet $y^3 - 172y^2 + 10800y - 360000 = 0$. Unde per inventionem diviſorum, prodibit $y = 100$. Dein habito $y$, radix ejus 10 erit $e$, $\frac{-86+100-60}{2}$ ſeu $-23$ erit $f$, & $+37$ erit $g$. Unde prodeunt æquationes quadraticæ $z^2 + 10z - 23 = 0$, & $z^2 - 10z + 37 = 0$, adeoque per reductionem, $z = -5 \pm 4\sqrt{3}$, & $z = +5 \pm 2\sqrt{-3}$. Ergo erit $x = (v + \frac{1}{4} = \frac{z+1}{4} =)$ $-1 \pm \sqrt{3}$, & $x = \frac{3 \pm \sqrt{-3}}{2}$, quæ erunt quatuor radices æquationis ſub initio propoſitæ $x^4 - x^3 -$ &c. $= 0$.



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230. Aliam methodum eruendi radices æquationum biquadraticarum mediantibus cubicis, aliquando concinniorem tradit *Euler*, quam ſic inveſtigare licet. Vide *Euler*, *Elem. Algebræ*, *Tom.* 1; aut *Martin*, *Elem. de Mathem. Toulouſe*, 1781.

Aſſumamus æquationem cubicam $y^3 - P'y^2 + Q'y - R' = 0$ cujus radices ſint $a$, $b$, $c$. Sitque $x = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Quadrando erit

$$x^2 = a + b + c + 2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})$$

Seu - $x^2 = P' + 2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})$

Adeoque $x^2 - P' = 2(\sqrt{ab} + \sqrt{ac} + \sqrt{bc})$

Et quadr. $x^4 - 2P'x^2 + P'^2 = 4(ab + ac + bc) + 8\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c})$

Seu - $x^4 - 2P'x^2 + P'^2 = 4Q' + 8\sqrt{R'}\,x$

Ergo - $x^4 - 2P'x^2 - 8\sqrt{R'}x + P'^2 - 4Q' = 0.$

Hujus itaque æquationis biquadraticæ una radix neceſſario erit $x = \sqrt{a} + \sqrt{b} + \sqrt{c}$, ut ex compoſitione manifeſtum eſt. Jam quo eruantur coefficientes æquationis cubicæ adſcititiæ $y^3 - P'y^2 + Q'y - R' = 0$, collatis inter ſe terminis homologis æquationis biquadraticæ noviſſime reſultantis & datæ $x^4 + Qx^2 + Rx + S = 0$, erit

$-2P' = Q$, adeoque $P' = -\frac{1}{2}Q$

$-8\sqrt{R'} = R$, adeoque $R' = +\frac{1}{64}R^2$

$P'^2 - 4Q' = S$, adeoque $Q' = \frac{1}{4}P'^2 - \frac{1}{4}S = \frac{1}{16}Q^2 - \frac{1}{4}S.$

Quibus valoribus $P'$, $Q'$, $R'$ ſubſtitutis, emerget æquatio cubica $y^3 + \frac{1}{2}Qy^2 + (\frac{1}{16}Q^2 - \frac{1}{4}S)y - \frac{1}{64}R^2 = 0$, cujus radices ſunt $a$, $b$, $c$, ut conſtat.

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Porro, quoniam ex tribus radicalibus $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, unaquæque poteſt eſſe vel affirmativa vel negativa, octo exorientur harum combinationes reſpectu ſignorum, quarum quatuor exponet $x$ ubi datæ æquationis terminus penultimus eſt affirmativus, reliquas ubi eſt negativus. Et quo diſtinguantur in utroque caſu combinationes idoneæ, obſervandum eſt, quod ſi R affirmativum ſit, ob $R = -8\sqrt{R'}$, erit $\sqrt{R'}$ ſive $\sqrt{a} \times \sqrt{b} \times \sqrt{c}$, negativum, unde neceſſe eſt ut ex tribus radicalibus una ſit negativa & reliquarum rectangulum affirmativum, quò contentum ſub omnibus evadat negativum. In hoc caſu igitur prodeunt 4 radices ſeu valores $x$,

$$x = \begin{cases} -\sqrt{a} + \sqrt{b} + \sqrt{c} \\ -\sqrt{a} - \sqrt{b} - \sqrt{c} \\ +\sqrt{a} + \sqrt{b} - \sqrt{c} \\ +\sqrt{a} - \sqrt{b} + \sqrt{c}. \end{cases}$$

Quod ſi R negativum ſit, erit $\sqrt{a} \times \sqrt{b} \times \sqrt{c}$ affirmativum, adeoque

$$x = \begin{cases} +\sqrt{a} + \sqrt{b} + \sqrt{c} \\ +\sqrt{a} - \sqrt{b} - \sqrt{c} \\ -\sqrt{a} + \sqrt{b} - \sqrt{c} \\ -\sqrt{a} - \sqrt{b} + \sqrt{c} \end{cases}$$

Ut ſi repetatur æquatio $z^4 * - 86z^2 + 600z - 851 = 0$, æquatio cubica $y^3 + \frac{1}{2}Q\,y^2 + (\frac{1}{16}Q^2 - \frac{1}{4}S)\,y - \frac{1}{64}R^2 = 0$, ſubſtitutis æquipollentibus fiet $y^3 - 43y^2 + 675y - 5625$

$= 0$, cujus reſolutio dabit $y = 25$, & dividendo per $y - 25 = 0$, eruetur quadratica $y^2 - 18y + 225 = 0$, cujus reductio dabit reliquas radices $y = 9 \pm 12\sqrt{-1}$. Dein extractis omnium radicibus quadraticis, nempe 5, necnon & $2\sqrt{3} \pm \sqrt{-3}$, per § 52; ob terminum penultimum $600z$ affirmativum, crit

$$z = \begin{cases} -5 + (2\sqrt{3} + \sqrt{-3}) + (2\sqrt{3} - \sqrt{-3}) = -5 + 4\sqrt{3} \\ -5 - (2\sqrt{3} + \sqrt{-3}) - (2\sqrt{3} - \sqrt{-3}) = -5 - 4\sqrt{3} \\ +5 + (2\sqrt{3} + \sqrt{-3}) - (2\sqrt{3} - \sqrt{-3}) = +5 + 2\sqrt{-3} \\ +5 - (2\sqrt{3} + \sqrt{-3}) + (2\sqrt{3} - \sqrt{-3}) = +5 - 2\sqrt{-3} \end{cases}$$

Ut antea. Ergo erit $x =$ &c. Cæterum facilius inventæ ſunt radices æquationis biquadraticæ propoſitæ per *Newtoni* methodum inveniendi diviſores, ſupra p. 176. Adeoque, non niſi deficientibus diviſoribus rationalibus ad haſce confugiendum.

Notandum reſtat, quod minuendo quadruplâ ratione radices æquationis cubicæ methodo *Carteſianâ* adſcitæ $y^3 + 2Qy^2 + (Q^2 - 4S)y - R^2 = 0$, immediate elicietur *Euleri* æquatio cubica $y^3 + \frac{1}{2}Qy^2 + (\frac{1}{16}Q^2 - \frac{1}{4}S)y - \frac{1}{64}R^2 = 0$.

Aliam inſuper methodum, & elegantem quidem (ut ſemper) educendi radices biquadraticarum mediantibus cubicis docet *Simpſon*, *Algeb.* p. 150, &c. ubi varia proponit artificia ad laborem calculi quibuſdam in caſibus particularibus minuendum. Verum de his ſatis.

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231. Methodus generalis reſolvendi æquationes quintæ altiorumve dimenſionum hactenus deſideratur; idque forſan haud magno analyſtarum detrimento, ſi modo ingentium calculorum in hiſce neceſſario inſtituendorum habeatur ratio. Pergamus itaque modos quamproxime eruendi radices æquationum irrationales docere. At priùs de inventione limitum inter quos conſiſtunt ſingulæ radices agendum.

## De limitibus radicum singularum æquationis.

232. Si in æquatione quacunque, binis quibuſdam numeris pro $x$ ſucceſſive ſubſtitutis, numeri reſultantes contrariis afficiantur ſignis, æquatio unam *ſaltem* radicem realem habebit inter hoſce numeros ſubſtitutos interjectam. Poteſt vero habere plures, quarum numerus ſemper erit impar.

233. Si in æquatione quacunque unam aut plures radices inæquales habente, ſucceſſive ſubſtituantur pro $x$ duo numeri quorum unus eſt major, alter minor unâ radicum realium, & quorum differentia ſimul ſit minor differentiâ inter hanc radicem & reliquarum proximam intercedente, numeri reſultantes neceſſario contrariis afficientur ſignis, & unica tantum radix realis inter hoſce numeros ſubſtitutos jacebit.

234. Quo eruatur autem numerus minimâ differentiarum inter binas quaſque radices intercedente minor; primo inveniatur æquatio quadratorum differentiarum radicum, $z^r - P'z^{r-1} + \&c. = 0$, per § 177. Dein ſub-

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ſtituendo $\frac{1}{u}$ pro $z$, inveniatur æquatio $u^r - P''u^{r-1} +$ &c. $= 0$, cujus radices erunt præcedentium reciprocæ. Quæratur limes radicum hujus affirmativarum per § 190, puta $l$, eritque $\frac{1}{l}$ minor radice minimâ affirmativâ $\frac{1}{u}$ ſeu $z$; adeoque $\frac{1}{\sqrt{l}}$ neceſſario erit minor quâlibet differentiarum radicum realium & inæqualium datæ æquationis. Et ſi ſumatur $\delta$ integer æqualis aut proxime major quam $\sqrt{l}$, *a fortiori*, erit $\frac{1}{\delta}$ minor quâlibet differentiarum, &c. Denique ſubſtituendo ſucceſſive in æquatione datâ terminos progreſſionis $0, \frac{1}{\delta}, \frac{2}{\delta}, \frac{3}{\delta}$, &c. pro $x$, numeri reſultantes tot exhibebunt variationes ſignorum quot habet æquatio radices reales & affirmativas, & unaquæque radicum inæqualium inter limites $\frac{m}{\delta}$ & $\frac{m+1}{\delta}$ jacebit, ſi modo numerorum reſultantium ſigna prodibunt contraria. Sive, quod eodem redit, & in praxi facilius erit, multiplicentur radices datæ æquationis per $\delta$, § 163, et in æquatione reſultante ſubſtituantur ſucceſſive termini progreſſionis 0, 1, 2, 3, &c. uſque ad extremos limites radicum affirmativarum. Et hoc pacto invenientur limites inter ſingulas æquationis reſultantis radices, qui utique per $\delta$ diviſi dabunt limites inter ſingulas radices inæquales affirmativas datæ æquationis. Et ſimiliter, limites inter ſingulas radices negativas invenire licet. Quod ſi termini ita ſubſtituti dant zero, erunt ipſæ radices æquationis.

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Cor. Hinc, ſi $\sqrt{l}$ prodeat unitate minor, erit $\frac{1}{\sqrt{l}}$ unitate major, adeoque æquatio aut unicam ſortietur radicem realem affirmativam, aut ſi plures, erunt binarum quarumque differentiæ unitate majores; & converſim: in hoc caſu uſurpare licet $\delta = 1$. Quòd ſi $\sqrt{l}$ prodeat major unitate, æquatio habere poteſt radices quarum differentiæ ſint unitate minores.

Hujus methodi auctor eſt *La Grange*, cæterum in æquationibus ſuperiorum dimenſionum parum utilis evadit, cum quantitatem $\frac{1}{\sqrt{l}}$ in hiſce eruere valde operoſum foret. In praxi commodior erit methodus inſequens.

235. Primo, minuantur radices datæ æquationis quantitate quadam cognitâ, § 156, & ſucceſſive ſubſtitutis pro $x$ terminis progreſſionis 0, 1, 2, &c. quærantur ex variatione ſignorum coefficientium novæ æquationis limites integri quorum differentia eſt 1. Dein, dupla ratione augeantur radices novæ æquationis, & quærantur limites radicum æquationis jam reſultantis quorum differentia eſt 1, nam inde innoteſcent limites arctiores radicum datæ æquationis, quorum nempe differentia eſt $\frac{1}{2}$. Atque iterum augeantur dupla ratione radices æquationis noviſſimæ, & quærantur hujus limites, inde vero innoteſcent limites radicum datæ æquationis adhuc arctiores, quorum nempe differentia eſt $\frac{1}{4}$. Et ſic deinceps.

Ut ſi proponatur æquatio $x^6 - 5x^5 + 2x^4 + 15x^3 - 10x^2 - x + 1 = 0$, erit

| | $x =$ | $+4$ | $+3$ | $+2$ | $+1$ | $0$ |
|---|---|---|---|---|---|---|
| $x^6 - 5x^5 + 2x^4 + 15x^3 - 10x^2 - x + 1$ | | + | — | + | + | + |
| $6x^5 - 25x^4 + 8x^3 + 45x^2 - 20x - 1$ | | + | + | — | + | — |
| $15x^4 - 50x^3 + 12x^2 + 45x - 20$ - - | | + | + | — | + | — |
| $20x^3 - 50x^2 + 8x + 15$ - - - | | + | + | — | — | + |
| $15x^2 - 25x + 2$ - - - - - - | | + | + | + | — | + |
| $6x - 5$ - - - - - - - - | | + | + | + | + | — |
| $1$ - - - - - - - - - - | | + | + | + | + | + |

Ubi obſervandum, quod ſubſtituto 4 pro $x$, coefficientes omnes novæ æquationis emergunt affirmativæ; ergo, + 4 erit maximæ radicis $a$ limes. Subſtituto autem + 3, prodit una variatio ſignorum; ergo maxima radix $a$ inter limites 4 & 3 conſiſtet. Et ſubſtituto + 2, prodeunt duæ variationes ſignorum, ergo ſecunda radix $b$, inter limites 3 & 2 conſiſtet. Et ſubſtituto + 1, prodeunt duæ variationes, ſubſtituto autem 0, quatuor; ergo tertia & quarta radix $c$ & $d$ inter limites 1 & 0 conſiſtent.

236. Jam quo eruantur limites radicum negativarum, mutatis ſignis terminorum alternorum, fiet

| | $x =$ | $0$ | $-1$ | $-2$ |
|---|---|---|---|---|
| $x^6 + 5x^5 + 2x^4 - 15x^3 - 10x^2 + x + 1$ | | + | — | + |
| $6x^5 + 25x^4 + 8x^3 - 45x^2 - 20x + 1$ - | | + | — | + |
| $15x^4 + 50x^3 + 12x^2 - 45x - 20$ - - | | — | + | + |
| $20x^3 + 50x^2 + 8x - 15$ - - - - - | | — | + | + |
| &c. | | + | + | + |

&c.

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Ubi obſervandum, quod ſubſtituto o pro $x$, prodeunt duæ ſignorum variationes, ſubſtituto autem + 1, una tantum; ergo radix $e$ inter limites o & — 1, conſiſtet. Et ſimiliter, radix $f$, inter limites — 1, & — 2.

237. Quod ſi deſiderentur limites radicum arctiores, multiplicando radices æquationis transformatæ per binarium, fiet,

| | +7 | +6 | +5 | +4 | +3 | +2 | +1 | 0 |
|---|---|---|---|---|---|---|---|---|
| $x^6 - 10x^5 + 8x^4 + 120x^3 - 160x^2 - 32x + 64$ | + | — | + | + | | + | — | + |
| $6x^5 - 50x^4 + 32x^3 + 360x^2 - 320x - 32$ - | + | + | — | — | | + | — | — |
| $15x^4 - 100x^3 + 48x^2 + 36cx - 160$ - - | + | + | — | — | | + | + | — |
| $20x^3 - 100x^2 + 32x + 120$ - - - - | + | + | + | — | | — | + | + |
| $15x^2 - 50x + 8$ - - - - - - | + | + | + | + | | — | — | + |
| $6x - 10$ - - - - - - - | + | + | + | + | | + | — | — |
| 1 - - - - - - - - - | + | + | + | + | | + | + | + |

Ex ſignorum reſultantium inſpectione conſtat, quod radix $2a$ conſiſtet inter limites 7 & 6, adeoque $a$ inter limites $3\frac{1}{2}$ & 3; $2b$ inter limites 6 & 5, adeoque $b$ inter limites 3 & $2\frac{1}{2}$; $2c$ inter limites 2 & 1, adeoque $c$ inter limites 1 & $\frac{1}{2}$; $2d$ inter limites 1 & o, adeoque $d$ inter limites $\frac{1}{2}$ & o. Et ſimiliter, radicum negativarum conſiſtet $e$ inter limites arctiores o & — $\frac{1}{2}$; & $f$ inter limites — $\frac{1}{2}$ & — 1. Et iiſdem veſtigiis inſiſtendo, invenire licet limites radicum adhuc arctiores. Prodeunt autem æquationis datæ radices,

| | | | |
|---|---|---|---|
| $a =$ | 3,303 | $e =$ | $-$ 0,303 |
| $b$ | 2,618 | $f$ | $-$ 1,618. |
| $c$ | 0,618 | | |
| $d$ | 0,383 | | |

Ut ex compoſitione manifeſtum erit.

Atque hic obſervandum eſt, quod inventis ſignis coefficientium æquationis transformatæ principis $x^6 - 5x^5$ &c. $= 0$, ex ſubſtitutione 0, 1, 2, 3, &c. ſucceſſive pro $x$, oriundis, ſimul determinantur ſigna coefficientium ſecundæ $x^6 - 10x^5$ &c. $= 0$, ex ſubſtitutione 0, 2, 4, 6, &c. oritura, utpote quæ neceſſario eadem erunt reſpective. Adeo ut hac methodo requiritur ſolummodo operatio ſubſtituendi in æquatione poſteriore numeros intermedios 1, 3, 5, &c.

238. Hac methodo inveniri poſſunt limites radicum realium inæqualium æquationis, non obſtantibus radicibus impoſſibilibus. Sic habitâ æquatione $x^4 - x^3 - 5x^2 + 12x - 6 = 0$, erit

| | $x$ | 2 | 1 | 0 |
|---|---|---|---|---|
| $x^4 - x^3 - 5x^2 + 12x - 6$ | | + | + | — |
| $4x^3 - 3x^2 - 10x + 12$ - - | | + | + | + |
| $6x^2 - 3x - 5$ - - - - - | | + | — | — |
| $4x - 1$ - - - - - - - | | + | + | — |
| 1 - - - - - - - - - | | + | + | + |

Ex hac operatione conſtat quod una ſaltem erit radix affirmativa inter 0 & 1, & quod duæ inter limites 1 & 2 conſiſtere poſſunt. Erit quoque

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| | $x$ | $0$ | $-1$ | $-2$ | $-3$ |
|---|---|---|---|---|---|
| $x^4 + x^3 - 5x^2 - 12x - 6$ | | $-$ | $-$ | $-$ | $+$ |
| $4x^3 + 3x^2 - 10x - 12$ - - | | $-$ | $+$ | $+$ | $+$ |
| $6x^2 + 3x - 5$ - - - - | | $-$ | $+$ | $+$ | $+$ |
| &c. | | $+$ | $+$ | $+$ | $+$ |

Unde conſtat, quod una ſaltem erit radix negativa inter limites $-2$ & $-3$, & quod tres eſſe poſſint affirmativæ; at hujus radices ſupra prodierunt $x = -1 \pm \sqrt{3}$, & $x = \frac{3 \pm \sqrt{-3}}{2}$: duæ igitur radices impoſſibiles in utroque caſu inter affirmativas latent. Hinc ſi operatione continuatâ quærantur limites arctiores, & inter limites binos ſucceſſivos, binæ ſemper exſurgent radices affirmativæ aut binæ negativæ, indicio eſt æquationem radices æquales aut impoſſibiles ſortiri.

239. Inventis hoc modo limitibus unius radicum inæqualium, facile determinari poteſt annon limes alteruter parte ſui $\frac{1}{10}$ a verâ radice differt. Primo enim, loco $x$ in æquationem ſubſtituatur limes inferior, iterumque hic limes parte ſui $\frac{1}{10}$ auctus; ſi numeri reſultantes prodeunt diverſis affecti ſignis, indicio eſt limitem inferiorem parte ſui $\frac{1}{10}$ a verâ radice non diſcrepare: contra, ſi numerorum reſultantium ſigna ſint eadem. Et ſimiliter, ſubſtituatur



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pro $x$ limes ſuperior, atque iterum hic limes parte ſui $\frac{1}{10}$ diminutus; ſi numeri reſultantes, &c. Hæc ita ſe habebunt ubi ipſorum limitum differentia major ſit parte $\frac{1}{10}$ limitis inferioris in priore caſu, ſuperioris autem in poſteriore; nam ſi minor ſit parte decimâ &c. *a fortiori*, erit differentia inter alterutrum limitem & radicem adhuc minor parte decima totius limitis.

## De resolutione æquationum per approximationem.

240. Reſolutionem æquationum numeralium per approximationem primus inſtituit *Vieta*, at methodo valde operoſa. Pars hæc doctrinæ æquationum in praxi utiliſſima, *Newtono* plurimum debet, qui duas tradidit methodas quamproxime eruendi radices æquationum numeralium in *Arithmetica Univerſali*, Cap. *De limitibus æquationum;* et in tractatu *De analyſi per æquationes infinitas*, Cap. 4. Harum priorem auctam promotamque jam proferemus.

241. Quoniam radicum ſingularum æquationis nullas radices impoſſibiles habentis quadrata ſunt affirmativa, quadratorum ſumma affirmativa erit, ideoque quadrato maximæ radicis major. Et eodem argumento, ſummæ biquadratorum, bicuborum pariumve deinceps poteſtatum radicum omnium tranſcendent poteſtates conſimiles radicis maximæ. Quoniam vero poteſtates altiores maximæ radicis longe majore ratione creſcunt quam conſimiles poteſ-

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tates reliquarum, (ſi modo inter radices ipſas notabile intercedat diſcrimen reſpectu magnitudinis) quo altius involvantur eo propius ad rationem æqualitatis inter ſe accedent poteſtates conſimiles maximæ radicis & ſummæ radicum omnium. Ut ſi æquatio ſit $x^3 - 19x + 30 = 0$, cujus radices ſunt 2, 3 & — 5; erit

$$5^2 : 5^2 + 3^2 + 2^2 :: 25 : 38 :: 1 : 1{,}52$$

$$5^4 : 5^4 + 3^4 + 2^4 :: 625 : 722 :: 1 : 1{,}1552$$

$$5^8 : 5^8 + 3^8 + 2^8 :: 390625 : 397442 :: 1 : 1{,}01745 \text{ \&c.}$$

$$5^{16} : 5^{16} + 3^{16} + 2^{16} :: 152587890625 : 152631002882$$
$$:: 1 : 1{,}000028 \text{ \&c.}$$

$$5^{32} : 5^{32} + 3^{32} + 2^{32} :: 1 : 1 + \frac{185302448381937}{23283064365386962890625} ::$$
$$:: 1 : 1{,}00000008, \text{ \&c.}$$

Unde elicitur methodus inſequens quamproxime eruendi radicem æquationis maximam ſeu a nihilo remotiſſimam.

242. *Quære ſummam quadratorum radicum & extrahe ejus radicem quadraticam; hæc enim radix major erit quam radix maxima æquationis. Sed ad radicem maximam propius accedes, ſi quæras ſummam biquadratorum & extrahas ejus radicem biquadraticam. Et adhuc propius, ſi quæras poteſtatem* 8$^{vam}$ *ſummæ radicum & extrahas ejus radicem* 8$^{vam}$. *Atque ita deinceps in infinitum.*

Sic in præcedente æquatione erit radix quadratica ſummæ quadratorum $\sqrt{38} = 6{,}164$&c. $\sqrt[4]{722} = 5{,}86$&c.

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$\sqrt[8]{397442} = 5{,}018$&c. $\sqrt[16]{152631002882} = 5{,}000028$&c. $\sqrt[32]{3328306\&c.} = 5{,}00000001$&c. Liquet igitur quod in æquationibus radices inæquales habentibus, ſurdæ radices ſummarum poteſtatum ſucceſſivarum radicum, decreſcendo ad verum valorem maximæ radicis cito convergunt, & quo major ſit inæqualitas radicum, eo citius.

243. Quod ſi duæ pluresve æquationis radices ſint fere inter ſe æquales, ita ut radix maxima cum alterâ ejuſdem circiter magnitudinis conſocietur, ſurdæ radices, &c. ſucceſſivæ ad verum valorem maximæ radicis tardius convergent; adeoque neceſſe erit ſummas altiorum poteſtatum radicum quærere, earumque ſurdas radices extrahere quo eruatur maximæ radicis valor quamproxime verus.

Ut ſi proponatur æquatio $8x^3 - 6x + 1 = 0$, ad tollendam coefficientem primi termini, poſito $\frac{1}{2}y$ pro $x$, exſurget $y^3 - 3y + 1 = 0$. Et poſito $Q = -3$ & $R = -1$, per § 170, prodit ſumma quadratorum radicum hujus $-2Q = 6$; ſumma biquadratorum $2Q^2 = 18$; ſumma 8 poteſt. $2Q(Q^3 - 4R^2) = 186$. At quoniam quadratum 6 ſeu 36 ſit ad 18 in ratione 2 ad 1; hujus vero quadratum ſeu 324 ad 186 in ratione paulo minore tantum, conſtat radices ſurdas ſummarum poteſtatum hactenus inventarum non niſi tarde ad verum valorem maximæ radicis convergere. Quæratur itaque ſumma 16 poteſtatum radicum $2Q^2(Q^6 - 24Q^3R^2 + 20R^4) = 25146$, cujus radix 16ma prodit 1,8837&c. Quæratur iterum ſumma 32 poteſtatum $2Q(Q^{15} - 112Q^{12}R^2 + 1144Q^9R^4 -$

$2112Q^6R^6 + 660Q^3R^8 - 16Q^{10}) = 6 \times 97945135$, hujus autem radix $32^{ma}$ prodit 1,87947&c. Conſentientibus igitur hiſce valoribus maximæ radicis ſolummodo in primo decimalium loco, colligere licet valorem ex ſumma 16 poteſtatum reſultantem, ultra primam decimalem 8 non eſſe verum. Adeoque ſi maximæ radicis valor accuratius deſideretur, neceſſe erit operationem continuando ſummam 64 poteſtatum radicum eruere, nempe $18 \times 1913111928$-2396037, cujus radix $64^{ma}$ prodit 1,87938530&c. Conſentientibus igitur hiſce valoribus in tribus decimalium locis, colligere licet radicem ſummæ 32 poteſtatum maximam radicem vere exponere ad uſque tertiam decimalem 9. Quanto propius ad veram radicem accedit radix ſummæ 64 poteſtatum, iiſdem veſtigiis inſiſtendo detegere licet; facilius vero methodo inſequenti.

244. Quoniam differentia radicum ſummæ 32 & 64 poteſtatum, nempe 0,0000847&c. prodit pars $22000^{ma}$ circiter poſterioris (cum hæc differentia toties ſaltem in 1,87938 &c. continetur) colligere licet priorem excedere maximam radicem parte ſui $22000^{ma}$ circiter. Unde, ſi tres radices deſignent $a$, $b$, $c$, quorum maxima ſit $a$, habetur $(a^{32} + b^{32} + c^{32})^{\frac{1}{32}} = a + \frac{1}{22000} a$. Et quoniam duæ radices in hac æquatione fere æquales neceſſario erunt $a$ & $b$, vel $b$ & $c$; ſit $1^{mo}$ $a$ ferme æqualis $b$. In hoc caſu, $b$ (ob defectum termini ſecundi) tantum exſuperabit $c$, ut $c^{32}$ quam minimum erit reſpectu $a^{32} + b^{32}$, adeoque tuto negligi poteſt. Erit igitur

$$(a^{32} + b^{32} *)^{\frac{1}{32}} = a + \frac{1}{22000} a$$

$$\text{Et } a^{32} + b^{32} = a^{32} + \frac{32}{22000} a^{32} + \frac{32 \cdot 31}{2 \cdot 22000^2} a^{32} + \&c.$$

$$\text{Itemque } b^{32} = \frac{32}{22000} a^{32} \text{ fere, } = \frac{2}{11 \times 5^3} a^{32}$$

$$\text{Adeoque } b^{64} = \frac{4}{11^2 \times 5^6} a^{64}$$

$$\text{Et } (a^{64} + b^{64})^{\frac{1}{64}} = \left(a^{64} + \frac{4}{11^2 \times 5^6} a^{64}\right)^{\frac{1}{64}}$$

$$= a\left(1 + \frac{4}{11^2 \times 5^6}\right)^{\frac{1}{64}} = a\left(\frac{1890629}{1890625}\right)^{\frac{1}{64}}$$

Unde liquet radicem $64^{am}$, 1,8793853 valorem maximæ radicis ad uſque $6^{tum}$ decimalium locum & ultra, vere deſignare.

$2^{do}$. Sit $b$ ferme æqualis $c$. In hoc caſu erit

$$(a^{32} + 2b^{32})^{\frac{1}{32}} = a + \frac{1}{22000} a$$

$$\text{Et } a^{32} + 2b^{32} = a^{32} + \frac{32}{22000} a^{32} + \&c.$$

$$\text{Itemque } 2b^{32} = \frac{32}{22000} a^{32} = \frac{2}{11 \times 5^3} a^{32}$$

$$\text{Adeoque } 4b^{64} = \frac{4}{11^2 \times 5^6} a^{64}, \ \& \ 2b^{64} = \frac{2}{11^2 \times 5^6} a^{64}$$

$$\text{Et } (a^{64} + 2b^{64})^{\frac{1}{64}} = \left(a^{64} + \frac{2}{11^2 \times 5^6} a^{64}\right)^{\frac{1}{64}} = a\left(\frac{1890627}{1890625}\right)^{\frac{1}{64}}$$

Adeoque in hoc caſu, *a fortiori* liquet, radicem ad uſque $6^{tum}$ decimalem & ultra, vere deſignari.

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Radici maximæ $y$ ſic inventæ 1,879385 &c. præponendum ſignum —, ſubſtituto enim — 1,879385 pro $y$, accedit $y^3 - 3y + 1$ propius ad nihil; nempe — 6,636293 &c. + 5,638155 &c. + 1 = 0,001 &c.

245. Inventâ maximâ radice, invenientur reliquæ, deprimendo æquationem cubicam ad quadraticam ope diviſoris inde reſultantis, cujus reductio dabit reliquas radices.

Aut ſimili methodo erui poteſt radix minima, inveniendo maximam radicem æquationis $z^3 - 3z^2 * + 1 = 0$, cujus radices ſunt præcedentium reciprocæ, ut patebit ſubſtituendo $\frac{1}{z}$ pro $y$; nam inventâ maximâ hujus radice $\frac{1}{c}$, per § 169, aut per formulas defectui tertii termini accommodandas, habetur $c = 0,347296$. Habitis igitur radicibus extremis, habetur media radix $b = 1,532088$ &c. utpote reliquarum complementum ad nihil. Ergo ob $x = \frac{1}{2}y$, prodeunt radices datæ æquationis $8x^3 - 6x + 1 = 0$,

— 0,939692 &c. + 0,766044 &c. + 0,173648 &c.

246. Ubi vero duæ pluresve æquationis radices ſunt æquales, haſce, ſi iiſdem ſignis afficiuntur, facilius eruere licet per methodum § 202, &c. Et ſi contrariis, licet per iſtam methodum immediate eruendæ non ſunt, tamen quærendo æquationem quadratorum radicum, quæ tot ſortietur radices æquales quot erant in datâ, atque ſignis affirmativis affectas, haſce per iſtam methodum eruere licet, et extractis harum radicibus quadraticis innoteſcent radices æquales datæ æquationis.



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247. Quod ſi ſummarum poteſtatum ſucceſſivarum hac methodo inveniendarum, prodit aliqua negativa, aut quadratum alicujus minus ſummâ poteſtatum proxime inſequente; indicio ẽſt æquationem radices impoſſibiles ſortiri. Veruntamen non obſtant radices impoſſibiles quo minus eruantur radices reales, ſi modo ex ſummis poteſtatum ſucceſſive reſultantibus quadratum cujuſque præcedentis ad inſequentem propius ſemper accedet.

Sic habita æquatione $x^3 - 5x^2 + 4x - 3 = 0$, quærendo æquationes quarum radices ſunt quadrata, biquadrata, &c. radicum datæ æquationis per § 169; ſummas quadratorum, biquadratorum, $8p$, $16p$, reſpective exponent 17, 317, 100709, 10142381189. Et cum $17^2 = 289$ prodit minus quam 317, indicio eſt æquationem radices impoſſibiles ſortiri; quoniam vero haud multùm deficit a 317, hujus verò quadratum 100489 propius adhuc accedit ad 100709, non neceſſe erit ſummas poteſtatum ultra quærere ad radicem proxime eruendam. Erit igitur

$$(100709)^{\frac{1}{8}} = 4{,}2195, \text{ et } (10142381189)^{\frac{1}{16}} = 4{,}220690;$$

quarum prior valorem radicis vere exprimet ad uſque primam decimalem 2 tantùm, poſterior vero ad uſque quintam 9 incluſive; ſicuti per methodum præcedentem colligere licet.

Habita quoque æquatione $y^3 - 2y - 5 = 0$, ſummas quadratorum, biquadratorum, $8p$, $16p$, $32p$, radicum exponent 4, 8, 432, 138912, 18833326272 reſpective. Et cum $8^2$ ſeu 64 prodit minus quam 432, indicio ẽſt æquationem radices impoſſibiles ſortiri; quoniam vero quadratum hujus 186624 jam proprius accedit ad 138912, at in-

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tervallo majore quidem quam in exemplo noviſſimo, extrahenda erit radix $32p$. Hæc autem prodit 2,09457 affirmative ponenda per § 148, quæ utique radicem ad uſque $4^{tam}$ decimalem 5, vere exprimet.

248. Hæc methodus radicibus maximis & minimis tantum æquationis educendis immediate inſervit, idque calculo ſatis operoſo, ubi duæ pluresve radices ſint inter ſe æquales aut accurate aut quamproxime; tardius nimirum ad verum radicis valorem convergentibus ſurdis radicibus ſummarum poteſtatum ſucceſſivarum: & interdum, perturbantibus radicibus impoſſibilibus, deficit. Methodus altera *Newtoni* jam explicanda præ cæteris ſimplex & uſui accommodata videtur: Æquationes in quibus aliqui vel nulli termini deſint, eâdam fere facilitate tractantur; et ſingulis æquationum radicibus educendis æque reſpondet.

## Methodus altera.

249. Sit æquatio $x^3 + ax^2 + bx + c = 0$ reſolvenda: inveniatur per § 239, aut per methodum præcedentem, $k$ limes qui minus quam parte ſui decimâ differt a radice quæſitâ. $1^{mo}$. Sit $p$ cum proprio ſigno hujus complementum ad verum radicis valorem, & ſubſtituto $k + p$ pro $x$ in æquationem, prodit per § 156, nova $(k^3 + ak^2 + bk + c) + (3k^2 + 2ak + b)\ p + (3k + a)\ p^2 + p^3 = 0$ (incipiendo nimirum a termino ultimo). Sive, deſignantibus A, B, C, D numeros coefficientes reſpective, $A + Bp + Cp^2 + Dp^3 = 0$. Et ſic univerſaliter: habitâ æquatione $x^n + ax^{n-1}$



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$+ bx^{n-2} + \&c. = 0$, ſubſtituendo $k + p$ pro $x$, erunt coefficientes novæ æquationis,

$$A = k^n + ak^{n-1} + bk^{n-2} + ck^{n-3} + \&c.$$
$$B = nk^{n-1} + n-1 . ak^{n-2} + n-2 . bk^{n-3} + \&c.$$
$$C = n . \frac{n-1}{2} k^{n-2} + n-1 . \frac{n-2}{2} ak^{n-3} + \&c.$$

&c. &c.

Æquationis igitur reſultantis $A + Bp + Cp^2 + Dp^3 + \&c. = 0$, radix $p$ exquirenda eſt, ut quotienti $k$ addatur; nempe (neglectis terminis $Cp^2$, $Dp^3$, &c. ob parvitatem) $A + Bp = 0$, ſive $p = \frac{-A}{B}$ fere. 2do. Sit $q$ cum proprio ſigno hujus complementum ad verum valorem $p$, & perinde ſubſtituto $\frac{-A}{B} + q$ pro $p$, in æquationem reſultantem, prodit nova $A' + B'q + C'q^2 + D'q^3 + \&c. = 0$, et neglectis terminis $C'q^2 + D'q^3 + \&c.$ ut antea, erit $q = \frac{-A'}{B'}$ fere. 3tio. Sit $r$ hujus complementum ad verum valorem $q$, & ſubſtituto $\frac{-A'}{B'} + r$ pro $q$, in æquationem ultimo reſultantem, prodit $A'' + B''r + C''r^2 + D''r^3 + \&c. = 0$, adeoque $r = \frac{-A''}{B''}$ fere. Et ſic operatione quouſque placuerit continuatâ, tandem emergit $x = k - \frac{A}{B} - \frac{A'}{B'} - \frac{A''}{B''} - \&c.$ quam proxime. Quo propius autem accedit $k$ ad verum radicis valorem, eo pauciora complementa adhibere opus erit ad radicem quamproxime eruendam.

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250. Emergente, per primam operationem, $p = \frac{-A}{B} - \frac{C}{B} p^2 - \frac{D}{B} p^3 -$ &c. quot ad loca decimalium primus hujus terminus $\frac{-A}{B}$ valorem $p$ fere deſignabit, cæteris neglectis; ſeu, quouſque continuanda eſt diviſio numeratoris A per denominatorem B, antequam reliqui termini $\frac{C}{B} p^2$, $\frac{D}{B} p^3$ valorem quotientis reſultantis afficiant; ſic inveſtigare licet.

Quoniam $p$, utpote per hypotheſin minus quam $\frac{1}{10} k$, eſt fractio decimalis unitate minor, ſi habeat cifras aliquas decimales primæ ejus figuræ ſignificanti præfixas, poteſtates ejus $p^2$, $p^3$, $p^4$, &c. ſaltem duplo, triplo, quadruplo, &c. plures cifras decimales præfixas habebunt reſpective. Adeoque 1$^{mo}$. Si coefficientes $\frac{C}{B}$, $\frac{D}{B}$, &c. ſint quoque decimales unitate minores, & quæ cifras aliquas decimales habeant præfixas, terminus quiſque $\frac{C}{B} \times p^2$, &c. tot ſaltem habebit cifras decimales præfixas quot ſunt in utroque factore ſimul. Et 2$^{do}$. Si coefficientes $\frac{C}{B}$, &c. ſint unitate majores, erit numerus cifrarum decimalium termino cuique $\frac{C}{B} p^2$, &c. præfixarum æqualis exceſſui cifrarum decimalium in $p^2$, &c. præ locis integris in $\frac{C}{B}$, &c. reſpective:

ut ex multiplicatione decimalium conſtat. Unde in omni caſu, cognito numero cifrarum decimalium complemento $p$ præfixarum, innoteſcet numerus cifrarum decimalium cuique terminorum inſequentium poſt primum, ſcil. $\frac{C}{B}p^2$, $\frac{D}{B}p^3$, &c. præfigendarum; atque inde regrediendo, facile conſtabit, quod primus terminus $\frac{-A}{B}$ valorem $p$ ad tot loca decimalium fere deſignabit, quot habet cifras decimales præfixas ſecundus terminus $\frac{C}{B}$ $p^2$, &c. Itemque, quod duorum aggregatum $\frac{-A}{B} - \frac{C}{B}p^2$ tot ad loca decimalium valorem $p$ fere deſignabit, quot habet cifras decimales præfixas tertius terminus $\frac{D}{B}$ $p^3$. Et ſic deinceps. Ac ſimiliter, determinare licet ad quot loca decimalium termini $\frac{-A'}{B'}$ & $\frac{-A''}{B''}$ complementa ſucceſſiva $q$ & $r$ reſpective, fere exponent.

251. Cæterum ſi quæras an valores modo inventi $\frac{-A}{B}$, $\frac{-A'}{B'}$, &c. veris valoribus $p$, $q$, &c. reſpective majores vel minores ſint; hoc inveniri poteſt, ſubſtituendo $\frac{-A}{B}$ pro $p$ in æquationem $A + Bp + Cp^2$ &c. $= 0$, nam ſi numerus reſultans affirmativus ſit, $\frac{-A}{B}$ major erit veritate, ſi negativus ſit, minor. Et ſic in cæteris. Idem quoque facilius

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innoteſcet ex operatione proximâ, ſubſtituendi nimirum $\frac{-A}{B} + q$ pro $p$; nam ſi ex æquatione reſultante prodeat $q$ affirmativus, indicio eſt quod $\frac{-A}{B}$ eſt minor, ſin $q$ ſit negativus, major quam verus valor $p$.

152. *Ex.* Repetatur æquatio $y^3 - 2y - 5 = 0$, per methodum priorem reducta, § 247. Et ſi ab initio auſpicari cum *Newtono* placuerit, inveniatur per § 239, limes inferior 2, qui minus quam decimâ ſui parte differt a radice quæſitâ. 1$^{mo}$. Subſtituatur $2 + p$ pro $y$ in æquationem, & prodibit nova $-1 + 10p + 6p^2 + p^3 = 0$, ſive (neglectis $6p^2 + p^3$ ob parvitatem) $-1 + 10p = 0$, et $p = \frac{1}{10} = 0{,}1$ fere. 2$^{do}$. Subſtituatur jam $0{,}1 + q$ pro $p$ in æquationem reſultantem, & emerget $+ 0{,}061 + 11{,}23q + 6{,}3q^2 + q^3 = 0$, ſive, neglectis, ut antea, terminis in quibus $q$ ultra primam dimenſionem aſſurgit, $q = \frac{-0{,}061}{11{,}23} = -0{,}0054$ fere (dividendo nempe ad uſque quatuor decimalium loca, cum $q$ duas habet cifras decimales primæ ſuæ figuræ ſignificanti 5 præfixas, adeoque $q^2$ necnon & $\frac{6{,}3q^2}{11{,}23}$ quatuor habebit, per § 250. Et proinde hic terminus $\frac{6{,}3q^2}{11{,}23}$ valorem $p$ ex ſecundâ operatione reſultantem non afficiet intra limites quatuor locorum decimalium.) 3$^{tio}$. Subſtituatur iterum $-0{,}0054 + r$ pro $q$ in æquationem noviſſimam, & emerget $0{,}000541550536 + 11{,}16204748r + 6{,}2838r^2$

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$+ r^3 = 0$, ſive $r = \frac{-0{,}000541550536}{11{,}16204748} = -0{,}00004851$ fere (dividendo nempe donec octo eliciantur decimalium loca, cum $r$ habet 4 cifras decimales primæ ſuæ figuræ ſignificanti 4 præfixas, adeoque $r^2$ necnon & $\frac{6{,}2838r^2}{11{,}16204748}$ octo habebit.) Prodit igitur

$$\left.\begin{matrix} k = +2 \\ p = +0{,}1 \end{matrix}\right\} = \quad - \quad - \qquad + 2{,}10000000$$

$$\left.\begin{matrix} q = -0{,}0054 \\ r = -0{,}00004851 \end{matrix}\right\} = \qquad - 0{,}00544851$$

$$\therefore y = \quad - \quad - \quad = \qquad + 2{,}09455149.$$

Ad minuendum ſubſtituendi laborem in tertia operatione, terminum altiſſimum $q^3$ æquationis ſecundò reſultantis propter exilitatem ſuam negligit *Newtonus*, unde ſubrepit error in ultimum figuram 3 valoris $r$ hoc pacto eliciti, 0,00004853. Minui quoque poteſt dividendi labor, quoniam quatuor tantum deſiderantur figuræ decimales ſignificantes in quotiente, abſcindendo figuras ſignificantes numeratoris ultra quatuor primas, itemque denominatoris (ob parvitatem figurarum ab initio) ultra quinque primas: erit enim $\frac{-0{,}0005415}{11{,}162} = -0{,}00004851$ ut antea.

Obſervandum eſt, quod qualibet operatione duplicantur figuræ decimales radici adjiciendæ. Atque in hac methodo ita ſemper ſe res habebit, niſi forſan ubi $\frac{C}{B}$ ſit uni-

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tate major, quo in caſu, figurarum radicis ultimarum aliquot per terminum $\frac{C}{B}p^2$ affici poſſunt. Quoad materiam, vix differt hæc methodus ab illâ quam tradidit *Ralphſon*, in *Analyſi Æquationum Univerſali*, an. 1690, quæ utique ſimiliter procedit aſſumendo duos terminos ultimos tantum æquationis ex quaque operatione reſultantis.

253. Accuratius autem eruetur valor $p$, aſſumendo tres terminos ultimos æquationis reſultantis, nempe $A + Bp + Cp^2 = 0$, ſeu $A + (B + Cp)\,p = 0$; unde prodit $p = \frac{-A}{B+Cp}$, ſeu, ſubſtituendo $\frac{-A}{B}$ in denominatorem pro $p$, $= \frac{-A}{B - \frac{CA}{B}}$, vel $= \frac{-AB}{B^2 - AC}$; quarum prior formula praxi magis accommodata videtur, re nimirum per unam multiplicationem & duas diviſiones peractâ, quæ in poſteriore per tres multiplicationes ac unam diviſionem procedit. Et hoc eſt theorema *rationale Halleianum*, *Phil. Tranſ.* an. 1694, N° 210 *. Aut aliquantò accuratius, per extractionem radicis ſurdæ quadraticæ, unde prodit

* *Analyſin per æquationes infinitas* excogitavit *Newtonus* ſaltem ante annum 1665. Vid. p. 33. Hanc a *Newtono* acceptam cum *Johanne Collins* communicavit *Barrow*, anno 1669. Eandemque in ſcriniis *Johannis Collins* repertam primus edidit *Jones*, an. 1711. Methodi *Newtonianæ* proxime eruendi radices æquationum numeralium & literalium, duobus terminis tantum æquationis reſultantis adhibitis, exemplum dedit *Wallis* in *Algebra*, an. 1683, *Anglice* editâ. De quo ita *Halley*, loc. cit. “ Qualia in hoc negotio [reſolvendi æquationes numerales] præſtiterit ſagaciſſima ingenii *Newtoniani* vis, ex contractiore ſpecimine a clariſſimo *Walliſio*, “ Cap. 94, Algebræ ſuæ edito, potius conjecturâ aſſequi quam pro certo comperiri licet. Ac dum obſtinata auctoris modeſtia amicorum precibus devicta “ cedat, inventaque hæc ſua pulcherrima in lucem promere dignetur, expectare “ cogimur.”

$p = \frac{-\frac{1}{2}B \pm \sqrt{\frac{1}{4}B^2 - AC}}{C}$, quarum ſeligenda eſt minor radix $\frac{-\frac{1}{2}B + \sqrt{\frac{1}{4}B^2 - AC}}{C}$, (in quâ utique partes $\frac{\frac{1}{2}B}{C}$ & $\frac{\sqrt{\frac{1}{4}B^2 - AC}}{C}$ contrariis ſignis afficiuntur) tum ob parvitatem ipſius $p$, tum quod hæc radix ad valorem prius inventum $\frac{-A}{B}$ propius accedit. Et hoc eſt theorema *irrationale Halleianum*, loco. cit. Sic in præcedente exemplo, per theorema *rationale*, prodit $p = \frac{1}{10 + \frac{6}{10}} = \frac{1}{10,6} = 0,0943$&c. et per theorema *irrationale*, prodit $p = \frac{-5 + \sqrt{31}}{6} = 0,9462$&c. quorum prior valor conſentit cum valore $p + q$, per *duas* operationes methodi præcedentis elicito, in duobus locis figurarum decimalium; poſterior autem, in tribus. Atque hinc forſan theorema *irrationale* tantùm, utpote accuratius, commendat *Newtonus:* utrumque tamen adhibere haud importunum erit, uno nimirum corrigente alterum.

254. Quoniam verò in hujuſmodi approximationibus maximi momenti eſt determinatio primæ figuræ decimalis complementi $p$, hanc quidem, tribus terminis ultimis adhibitis ſemper explorare convenit ubi in æquatione $A + Bp + Cp^2$ &c. $= 0$, quadratum coefficientis penultimi termini $B^2$ non fit decies majus quam factus ex ultimo termino $A$ in coefficientem termini antepenultimi $C$.

Nam quoniam valor $p$, adhibitis duobus terminis ultimis tantum æquationis reſultantis, fit $\frac{-A}{B}$ fere; adhi-

bitis autem tribus, fit $\frac{-AB}{B^2-AC}$ accuratius, $= \frac{-A}{B} - \frac{A^2C}{B^3}$ circiter: (ut patebit dividendo — AB per B² — AC) differentiam horum valorum fere exponet $\frac{-A^2C}{B^3} = \frac{AC}{B^2} \times \frac{-A}{B}$. Unde poſito $\frac{AC}{B^2} = \frac{1}{10}$, differentia valorum evadet $\frac{1}{10} \times \frac{-A}{B}$ fere, hoc eſt, pars decima fere totius valoris prioris; poſito $\frac{AC}{B^2} = \frac{1}{9}$, evadet major parte decimâ totius; poſito $\frac{AC}{B^2} = \frac{1}{8}$, evadet adhuc major parte decimâ totius: et ſic deinceps, creſcente valore $\frac{AC}{B^2}$, ſeu quod perinde eſt, decreſcente ratione B² ad AC, eo magis recedet prior valor $\frac{-A}{B}$ a poſteriore $\frac{-AC}{B^2-AC}$, ſive, *a fortiore*, a valore accuratiore $\frac{-\frac{1}{2}B + \sqrt{\frac{1}{4}B^2 - AC}}{C}$. Hinc, ſi B² non ſit decies majus quam AC, differentia valorum tanta eſſe poteſt, quanta errorem in ipſam primam figuram decimalem prioris valoris $\frac{-A}{B}$ inducat. Quod ſi B² ſit decies aut plus quam decies majus quam AC, differentia iſta figuras decimales in $\frac{-A}{B}$ poſt primum locum tantùm afficiet. Et ſimiliter, ſi B² ſit centies majus quam AC, figuras deci-

males poſt ſecundum locum tantum; ſi millies, poſt tertium locum tantum afficiet. Atque ita in cæteris $q$, $r$, $s$, &c.

Obſervandum eſt, quod ope theorematis *irrationalis* qualibet vice triplicatur numerus locorum decimalium radici per præcedentem operationem adjiciendorum, per § 250. Hoc pacto igitur figuras decimales duplo plures quam per methodum præcedentem, qualibet vice radici lucraberis.

255. Hinc ſi radicem ad uſque quintum aut ſextum decimalium locum tantum quæras, in praxi plerumque facilius erit utramque methodum adhibere. Eruendo nimirum valorem primi complementi $p$ ex tribus ultimis terminis æquationis reſultantis; deinde *Halleium* ſecuti, opus de integro repetere, ſubſtituendo in æquationem datam, valorem radicis jam inventum $+ q$ pro $x$; (præſertim ubi aliquis terminus datæ æquationis deeſt, hoc pacto enim minuitur ſubſtituendi opus) unde ex binis terminis æquationis ultimò reſultantis habebitur $q$. Sic ex conſenſu theorematis *rationalis* & *irrationalis* prodiit $p = ,094$: deinde ſubſtituendo in æquationem datam $2,094 + q$ pro $x$, prodit nova $- 0,006153416 + 11,154508q + \&c. = 0$, adeoque $q = + \frac{0,006153416}{11,154508} = + 0,000551$. Adeoque $x = 2,094 + q = 2,094551$.

256. Quod ſi radix ad plurima decimalium loca deſideratur quam proxime, laborem plerunque minues, præſertim in æquationibus ſuperiorum dimenſionum, ſi ſingula complementa $p$, $q$, $r$, &c. extrahendo minorem radicum ex tribus ultimis terminis æquationis noviſſimè reſultantis exquiras.

*Ex.* 2. Proponatur æquatio $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$, cujus radicem in ſolutionem diffi-

ciſſimi problematis Arithmetici, *Vietæ* methodo, accuratiſſime aſſecutus eſt *Wallis*, *Algeb.* Cap. 62; quamque itidem methodi ſuæ exemplum affert *Ralphſon*, p. 25, 26. Ejuſmodi autem formæ eſt hæc æquatio, ut plures habeat radices affirmativas, & quod reſolvendi difficultatem adaugeat, prægrandes ſunt coefficientes ejus reſpectu termini ultimi.

Minuendi gratiâ, dividantur hujus radices per denarium, & prodibit $y^4 - 8y^3 + 20y^2 - 15y + 0{,}5 = 0$. Inveniatur limes hujus $k = 1$, & ſubſtituendo $1 + p$ pro $y$, prodibit nova $-1{,}5 + 5p + 2p^2 -$ &c. $= 0$, adeoque per extractionem radicis minoris, erit $p = \frac{-5 + \sqrt{37}}{4} = 0{,}27$. & $y = 1 + p = 1{,}27$: adeoque $x = 10y = 12{,}7$. Subſtituatur iterum $12{,}7 + q$ pro $x$, in æquationem datam, & prodibit $-298{,}6559 + 5296{,}132q - 82{,}26q^2 -$ &c. $= 0$, adeoque $q = \frac{2648{,}066 - \sqrt{6987686{,}106022}}{82{,}26} = 0{,}05644080$ &c. Per theorema *rationale* quoque prodibit $q =$

$$= \frac{298{,}6559}{5296{,}132 - \left(\frac{82{,}26 \times 298{,}6559}{5296{,}132}\right)} = \frac{298{,}6559}{5296{,}132 - 4{,}63875} =$$

$0{,}05644075$ &c. Unde, his valoribus collatis, certi eſſe poſſumus $q$ fere deſignari ad uſque quinque decimalium loca ſaltem; adeoque erit $x = 12{,}7 + q = 12{,}75644$ proxime. Et ſi plures deſiderentur radicis figuræ, de integro repetatur calculus, ſubſtituendo $12{,}75644 + r$ pro $x$ in æquationem datam, & aſſumptis tribus terminis æquationis reſultantis, prodibit $r = 0{,}0000017944807 44$: adeoque radix accuratiſſima fit $12{,}75644179448074 4$. redintegratione calculi nimirum triplicante numerum figurarum decimalium prius inventarum.

257. Labor hujus methodi præcipuus in operatione ſubſtituendi quantitates unas pro aliis reperietur: id quod varie perficias, ſequentem modum maxime expeditum cenſet *Newtonus*, præſertim ubi numeri coefficientes conſtant ex pluribus figuris.

Repetatur æquatio præcedens $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$. Reſolvi hæc poteſt in factores hujus formæ,

$$\overline{\overline{\overline{\overline{(x - 80} \times x} + 1998} \times x} - 14937 \times x + 5000 = 0.$$

Ubi prodit primus $x - 80$; ſecundus, $x$; tertius, factus ex præcedentibus una cum $1998$; quartus, $x$; quintus, factus ex præcedentibus una cum $-14937$; ſextus, $x$; et facto ſub omnibus adjiciendus $5000$. Subſtituendo autem in ſingulis factoribus $12{,}7 + q$ pro $x$, erit

I. $- 67{,}3 + q$
$\times \; 12{,}7 + q$

II. $- 854{,}71 - 54{,}6q + q^2$
$+ 1998$

III. $+ 1143{,}29 - 54{,}6q + q^2$
$\times \; 12{,}7 + q$

IV. $+ 14519{,}783 + 449{,}87q - 41{,}9q^2 +$ &c.
$- 14937$

V. $- 417{,}217 + 449{,}87q - 41{,}9q^2$
$\times \; 12{,}7 + q$

VI. $- 5298{,}6559 + 5296{,}132q - 82{,}26q^2 -$ &c.
$+ 5000$

VII. $- 298{,}6559 + 5296{,}132q - 82{,}26q^2 -$ &c. $= 0$.

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258. Hæc methodus radicibus æquationum tam purarum quam affectarum educendis inſervit, & proinde in extrahendis radicibus numerorum altioribus utilis erit. Vide *Halley*, loco. cit. Porro, inventâ quotiente $k$, $p$, &c. æquatio ſemper relinquitur cujus radix unà cum acquiſitâ quotiente adæquat radicem æquationis primo propoſitæ. Unde examinatio operis hic æque poterit inſtitui ac in reliquâ Arithmeticâ, auferendo nempe quotientem a radice primæ æquationis (ſicut analyſtis notum eſt) ut æquatio ultima, vel termini ejus duo treſve ultimi, inde producantur. Sic invento $p = 0{,}1$, erit $k + p = 2{,}1$, adeoque ſubſtituendo $2{,}1 + q$ pro $y$, in æquationem primam $y^3 - 2y - 5 = 0$, prodibit $0{,}061 + 11{,}23q + 6{,}3q^2 + q^3 = 0$; ipſa nempe æquatio quæ ex ſubſtitutione $0{,}1 + q$ pro $p$ in æquationem primo reſultantem prodiit: atque ita iſtius operationis veritas comprobatur.

259. Ex dictis facile judicabit lector, an valent multi nominis viri *La Grange* objectiones, *Differt. Acad. Berlin*, an. 1768. nempe, quod methodo *Newtonianâ* ignoratur quantum ad veritatem accedit tum valor ($k$) primo aſſumptus, tum valor cujuſque complementi inſequentis, ($p$), ($q$), &c. Methodus certe, quam ut accuratiorem commendat, *Newtonianæ* facilitate longe cedit, quippe quæ ex inventione quantitatis $\sqrt{l}$, § 234. non niſi per tentamina valde laborioſa in æquationibus plurimarum dimenſionum eruendæ, auſpicatur; per inventionem limitum æquationum ex binomiis compluribus ſucceſſive ſubſtitutis reſultantium, procedit; et per combinationem plurium fractionum continuarum ex hiſce operationibus elicitarum, concludit.

260. Ad evitandum tertiæ ſubſtitutionis laborem in exemplo ſecundo, correctiones duas ſatis artificioſas adhibet *Halley*, quarum priore, notas veras in aſſumptâ ($k$) quintu-

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plices reddit; poſteriore autem, etiam duplum numerum cifrarum adjungit: ut omnino aſſumptas ſeptuplicet. Cæterum iteratâ hâc operatione parùm promovetur calculus. Simili methodo inſiſtit *Bernoulli*, *Lect. Calculi integ.* Lect. 53.

Poſſimus quoque *Le Marquis de Courtivron* atque *Eulerum* ſecuti, valorem $p$ ex pluribus terminis æquationis reſultantis elicere, ſubſtituendo nimirum in hanc æquationem valores $p^2$, $p^3$, &c. ex quaque præcedente operatione inventos, ſcil. ſubſtituendo $\frac{A^2}{B^2}$ pro $p^2$; deinde, ſubſtituendo $\left(-\frac{A}{B}-\frac{C}{B}\times\frac{A^2}{B^2}\right)^3$ pro $p^3$; & ſic deinceps. Unde elicitur formula inſequens, $p=$

$$-\frac{A}{B}-\frac{C}{B}\left(\frac{A}{B}\right)^2-\left(\frac{2C^2-BD}{B^2}\right)\left(\frac{A}{B}\right)^3-\left(\frac{5C^3-5BCD+B^2E}{B^3}\right)\left(\frac{A}{B}\right)^4-$$

&c. Sic repetito exemplo 1^mo^. prodit $\frac{A}{B}=-\frac{1}{10}=-0{,}1$. $\frac{A^2}{B^2}=0{,}01$. $\frac{A^3}{B^3}=-0{,}001$. $\frac{A^4}{B^4}=0{,}0001$. $\frac{C}{B}=0{,}6$. $\frac{2C^2-BD}{B^2}=0{,}62$. $\frac{5C^3-5BCD}{B^3}=0{,}78$. Unde prodit $p=+0{,}1-0{,}006+0{,}00062-0{,}000078=0{,}94542$. Vide *Mem. De l'Acad. Paris*, 1744, p. 405. Præſtat autem methodus *Newtoniana*, utpote quæ iteratâ operatione ad verum radicis valorem citius convergit.

Aliam approximandi methodum tradit *Taylor*, ex ipſius *Incrementorum* theoriâ petitam, *Phil. Tranſ.* an. 1717. Duas quoque, & egregias quidem, excogitavit *Simpſon*, quarum prior per ſeriem infinitam citiſſime convergentem procedit, *Mathematical Diſſertations*, p. 102; poſterior verò ex calculo *fluxionali* pendet, *Eſſays on various ſubjects*, p. 81.

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